Complex Variables II
Assignment 8

1. The *implicit function theorem* states that a relation $F(w,z)=0$ can "implicitly" determine z as a function of w. Here is a version of this general principle specialized to the case with F being a complex analytic function of complex variables z and w. We use subscripts to denote partial derivatives, so

$$F_w(w,z) = \frac{\partial F(w,z)}{\partial w} \;, \qquad F_z(w,z) = \frac{\partial F(w,z)}{\partial z} \;, \text{ etc.}$$

These are complex derivatives as usual in complex analysis.

The equation $F(w,z)=0$ defines a "curve" in the complex "plane" $\mathbb{C}^2$ that by tradition in this context may be called X. Suppose $(w_0,z_0)\in X$ (i.e., $F(w_0,z_0)=0$). Is it possible to solve for w near w_0 for z near z_0? That would be to find an open set $\Omega\subseteq\mathbb{C}$ with $z_0\in\Omega$ and an analytic function g defined on Ω so that $w_0=g(z_0)$ and if $w=g(z)$ then $F(w,z)=0$ (i.e., $F(g(z),z)=0$). Is this solution locally unique? That would be that there is an $r>0$ so that if $|w'-w|<r$ and $F(w',z)=0$, then $w'=w$.

We can study these questions using informal calculus. Suppose $z=z_0+\Delta z$ and $w=w_0+\Delta w$. Then (if F is C^2 in the appropriate sense)

$$\begin{aligned} F(w,z) &= F(w_0+\Delta w, z_0+\Delta z) \\ &= F_w(w_0,z_0)\Delta w + F_z(w_0,z_0)\Delta z + O\Big(\left|\Delta w\right|^2 + \left|\Delta z\right|^2 \Big) \;. \end{aligned}$$

Setting $F(w,z)=0$ and ignoring the error term gives

$$\Delta w \approx \frac{F_z(w_0,z_0)}{F_w(w_0,z_0)}\,\Delta z \;.$$

This suggests that Δw exists and is uniquely determined by Δz if

$$F_w(w_0,z_0) \neq 0 \;. \qquad (1)$$

This exercise proves an analytic implicit function theorem under the nondegeneracy hypothesis (1) and the regularity hypothesis that F is locally C_2, which means that all partial derivatives of F up to second order exist and are continuous functions of w and z in a neighborhood of (w_0,z_0) in $\mathbb{C}^2$. These are complex partial derivatives of the kind that imply that F is analytic in w for any fixed z and analytic in z for any fixed w. The proof is close to the proof of the inverse function theorem from an earlier assignment.

(a) (*uniqueness*) If z, w and w' are close enough to z_0 and w_0 respectively, and if $F(w,z)=0$ and $F(w',z)=0$, then $w=w'$. *Hint.*

$$F(w',z) = F(w,z) + F_w(w,z)(w'-w) + O\Big(|w'-w|^2\Big) \ .$$

(b) (*existence*) Let $\gamma(t)$ be a contour in the w variable of the form $\gamma(t) = w_0 + re^{it}$, for $r>0$ but sufficiently small. Define

$$h(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{F(u,z)}\, du \ .$$

Show that $h(z)$ is a continuous function of z with $h(z_0)\neq 0$. Show that for z close enough to z_0, there is at least one w close to w_0 so that $F(w,z)=0$.

(c) (*lemma for analyticity*) Suppose $H(w,z)$ is C_2 in the sense above with w in a neighborhood of γ and z in a neighborhood of z_0. Define

$$\phi(z) = \int_\gamma H(u,z)\, du \ .$$

Show that $\phi(z)$ is a differentiable function of z in a neighborhood of z_0.

(d) (*analyticity*) Define

$$g(z) = \frac{1}{2\pi i}\int_\gamma \frac{uF_w(u,z)}{F(u,z)} du \ . \tag{2}$$

Show that $w=g(z)$ is the locally defined implicit function. Specifically: $F(g(z),z)=0$, $g(z_0)=w_0$, and g in defined and analytic in a neighborhood of z_0.

(e) (*for Exercise 2*) Verify the implicit differentiation formula that motivated the necessary condition (1):

$$g'(z) = \frac{dw}{dz} = \frac{F_z(w,z)}{F_w(w,z)} \ .$$

2. Suppose $X\subset\mathbb{C}^2$ is defined by $F(w,z)=0$, where F is a polynomial in w and z. If $p_0=(w_0,z_0)\in X$ and $p_1=(w_1,z_1)\in X$, define the euclidean distance as points in $\mathbb{C}^2\sim\mathbb{R}^4$:

$$d(p_0,p_1) = |p_1-p_0| = \Big(|w_1-w_0|^2 + |z_1-z_0|^2\Big)^{\frac{1}{2}} \ .$$

This makes X a metric space. Define the complex gradient by $\nabla F(w,z) = (F_w(w,z), F_z(w,z))$. Suppose that $\nabla F(w,z)\neq 0$ if $(w,z)\in X$. Show that X is an abstract Riemann surface as defined in class. A *local coordinate* is a one to one analytic mapping $\xi \rightsquigarrow \phi(\xi) = (w(\xi), z(\xi)) \in X$ from an open set $\Omega_\phi \subseteq \mathbb{C}$ to an open set $\phi(\Omega) = N_\phi \subseteq X$. Let $\psi : \eta \rightsquigarrow \psi(\eta) = (w(\eta), z(\eta))$ be another local coordinate that "overlaps" ξ in the sense that N_ϕ intersects N_ψ. We say that the local coordinates are *compatible* if the composite $\eta = \psi^{-1}(\phi(\xi))$ is a complex differentiable function where it is defined. We say that z is a local coordinate near $p_0 = (w_0, z_0)$ if $F_w(w_0, z_0) \neq 0$ and the map $z \rightsquigarrow \phi(z) = (g(z), z) \in X$ is defined by Exercise 1.

(a) Show that if $p_0 \in X$, then either z or w (or both) is a local coordinate near p_0.

(b) Show that if $F_w(w_0, z_0) \neq 0$ and $F_z(w_0, z_0) \neq 0$ then the local coordinates $z \rightsquigarrow (g(z), z)$ and $w \rightsquigarrow (w, h(w))$ (the latter also defined by the implicit function theorem) are compatible. *Hint.* Part (e) of Exercise 1 and the inverse function theorem are relevant.

(c) Show that if z is a local coordinate near p_0 and p_1, and if the neighborhoods overlap, then the local coordinates are compatible.

(d) Show that X is an abstract Riemann surface (a "complex curve") in the sense that every point $p \in X$ has a local coordinate and any two local coordinates are compatible if their images overlap.

3. There are connections between complex analysis, algebraic geometry, and number theory. This is particularly true in the study of Diophantine equations and corresponding algebraic "varieties". A *Diophantine* equation is a polynomial relation $f(x_1, \cdots, x_n) = 0$, where f is a polynomial in n variables with integer coefficients, where we seek solutions with all x_k being integers. The basic example is $f(x, y, r) = x^2 + y^2 - r^2$ (a quadratic polynomial, also called a quadratic *form*, in three variables $(x, y, r) = (x_1, x_2, x_3)$). Integers x, y, r with $x^2 + y^2 = r^2$ are *Pythagorean triples*. Geometrically, a Pythagorean triple represents a right triangle where all the sides have integer length.

   You can divide the relation by r and get an equivalent equation involving $z = \frac{x}{r}$, and $w = \frac{y}{r}$, which is

$$z^2 + w^2 = 1 \ , \quad z, w \text{ rational} \ . \tag{3}$$

   If we take z and w to be complex variables, this becomes the equation of an "curve" in $X \subset \mathbb{C}^2$. The problem of finding Pythagorean triples may be formulated as the problem of finding *rational points* on the Pythagorean curve X. That is, points $(z, w) \in \mathbb{C}^2$ which are rational and $(z, w) \in X$. It turns out that understanding X as a complex curve leads to a recipe for all Pythagorean triples. This comes from a famous construction that shows that if you add a point at infinity, then X is equivalent to the Riemann sphere.

   (a) Consider the point $(-1, 0) \in X$. For each number t, consider the line that connects $(-1, 0)$ to $(0, t)$. This is a line with slope t.

Show that this line intersects X at exactly one point other than the "base point" $(-1, 0)$. Call that point $F(t)$. *Remark*, you can think of this as "obvious" because the line has the "point/slope" form $\{(z, w) \text{ with } w = 0 + t(z + 1)\}$. The point is in X if u satisfies a quadratic equation, whose coefficients depend on t, with one solution $z = -1$. The point you seek is the other solution.

(b) Take $t = \frac{a}{b}$ to be a rational (ratio of integers a and b), clear denominators in the formulas from part (a) and get a recipe for Pythagorean triples. Experiment with small a and b to get a few *primitive* triples (triples that are not integer multiples of other triples, i.e., x, y, and r have no common factors).

(c) Let t be a point in the Riemann sphere, S. Show that the map F from part (a) defines a one to one and onto map from S to X, if $\infty \in S$ is mapped to $(-1, 0) \in X$.

(d) (*Not to hand in because it's lot of not very interesting "routine verification", but you should be aware that it's possible*) Show that F defines an analytic equivalence of the Riemann surfaces X and S. That is, if $p \in X$ maps to $q = F(p) \in S$, and if ξ is a local coordinate near p and η is a local coordinate near q, then the map $\xi \rightsquigarrow \eta$ is an analytic function.