Complex Variables II
Assignment 4

Correction. Exercise 3b and 3c corrected to $S(z)$ instead of $P(z)$.

1. (*This is two conclusions from the proof that started in class.*) Let $f(z)$ be an analytic function defined in a neighborhood of $z = 0$. Suppose $f(0) = 0$ and $f'(0) = 0$. The power series of f about zero converges in a neighborhood of zero:
$$f(z) = \sum_{k=0}^{\infty} a_k z^k \ .$$
Let n be the smallest integer with $a_n \neq 0$, so f has a zero of order n at zero.

   (a) Show that f is not one to one in any neighborhood of zero. More precisely, show that there is an $r > 0$ so that for any positive ρ with $\rho < r$ there are $z_1 \neq z_2$ with $f(z_1) = f(z_2)$. *Hint.* Let $\gamma(t) = re^{it}$ be the radius r contour and consider the winding number
   $$D(w) = \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta) - w} d\zeta \ .$$
   Then $D(w) = n$ for w close enough to zero and roots of $f(z) = w$ must be simple if is small enough and $w \neq 0$.

   (b) (*Open Mapping Theorem*) Let $\Omega \subseteq \mathbb{C}$ be an open set and f an analytic function defined on Ω. Show that the image of Ω under f is an open set. The image is
   $$\Omega' = f(\Omega) = \{f(z) \text{ with } z \in \Omega\} \ .$$
   Hint. Ω' is open if every $w_0 \in \Omega'$ has an $r > 0$ so that if $|w - w_0| \leq r$ then $w \in \Omega'$. If $w_0 = f(z_0)$ and $f'(z_0) \neq 0$, this is a consequence of the local inverse function theorem. See part (a) for the case $f'(z_0) = 0$.

2. (*Fourier series*) Let $f(z)$ be an analytic function defined in a domain Ω that contains the unit circle. This implies that there is $R > 1$ so that f is analytic in the closed annulus A defined by $\frac{1}{R} \leq |z| \leq R$. Define three circular contours $\gamma_- = \left\{|z| = \frac{1}{R}\right\}$, $\gamma_0 = \{|z| = 1\}$, and $\gamma_+ = \{|z| = R\}$.

   (a) Show that if z is in the interior of A then
   $$f(z) = g(z) - h(z) \ , \ \ g(z) = \frac{1}{2\pi i}\int_{\gamma_+} \frac{f(\zeta)}{\zeta - z} d\zeta \ , \ \ h(z) = \frac{1}{2\pi i}\int_{\gamma_-} \frac{f(\zeta)}{\zeta - z} d\zeta \ , \ .$$

(b) Show that g is defined and analytic in the disk $|z| < R$ and has Taylor expansion (note: the contour is γ_0)

$$g(z) = \sum_{n=1}^{\infty} a_n z^n \ , \ \ a_n = \frac{1}{2\pi i} \int_{\gamma_0} f(\zeta) \zeta^{-(n+1)} d\zeta \ .$$

Show that for real θ, g has the *Fourier series* representation

$$g(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \ , \ \ a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) d\theta \ .$$

[The integral formulas are also true with g in place of f and may seem more natural that way, but we eventually need them with f.]

(c) Show that h is analytic outside the smaller disk $|z| > \frac{1}{R}$ and had the *Laurent expansion*. [We use m in the first formula for simplicity. We switch to n in the second formula to make it fit with the formula from part (b).]

$$h(z) = \sum_{m=0}^{\infty} b_m z^{-m-1} \ , \ \ b_m = \frac{-1}{2\pi i} \int_{\gamma_0} f(\zeta) \zeta^m d\zeta \ .$$

Show that the sum converges "geometrically", which means there are positive constants C_1 and C_2 with $\left| b_m z^{-m-1} \right| \leq C_1 e^{-C_2 m}$ if $|z| \geq 1$. Show that for real θ, h has the Fourier series representation

$$h(e^{i\theta}) = - \sum_{n=-\infty}^{-1} a_n e^{in\theta} \ , \ \ a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) d\theta \ .$$

(d) Show that if f is analytic in a neighborhood of the unit circle, then

$$f(e^{i\theta}) = \sum_{-\infty}^{\infty} a_n e^{in\theta} \ , \ \ a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta} d\theta \ .$$

Show that the sum converges geometrically in the sense that $|a_n| \leq C_1 e^{-C_2 |n|}$.

(e) Show that if $\phi(t)$ is real analytic for real t and periodic with period 2π, which means $\phi(t + 2\pi) = \phi(t)$ for all real t, then ϕ has a Fourier series representation of the form

$$\phi(t) = \sum_{-\infty}^{\infty} a_n e^{int} \ , \ \ a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \phi(t) dt \ .$$

3. The Riemann sphere may be identified with the unit sphere in 3D. Let (x, y) be a point in the plane and $(x, y, 0)$ the corresponding point in 3D. The line connecting $(x, y, 0)$ to the "north pole" $(0, 0, 1)$ intersects the unit sphere at a unique point (ξ, η, ζ). The plane is identified with $\mathbb{C}$ with $(x, y) \longleftrightarrow z = x + iy$, so $|z|^2 = x^2 + y^2$.

(a) Show that this point is given by the *stereographic projection* formulas

$$S(z) = (\ \xi,\ \eta,\ \zeta\ ) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

(b) Show that $z_n \to z$ as $n \to \infty$ if and only if $S(z_n) \to S(z)$.

(c) Show that $|z_n| \to \infty$ as $n \to \infty$ if and only if $S(z_n) \to (0,0,1)$ (the north pole).

(d) Let $f(z) = \frac{az+b}{cz+d}$ be a fractional linear transformation. Define the corresponding map G on the Riemann sphere, $Q = G(P)$, by

$$P \to Q = G(P) \ \text{ if } \ P = S(z) \ \text{ and } \ Q = S(f(z)) \ .$$

The map G may be expressed as $G = S \circ f \circ S^{-1}$ because it is formed by first going from the sphere to the plane, then using f, then going back to the sphere:

$$P \longrightarrow S^{-1}(P) \longrightarrow f(S^{-1}(P)) \longrightarrow S(f(S^{-1}(P))) \ .$$

Show that G is a continuous map for all P (including the north pole) and for any fractional linear transformation. Show that G is one to one and onto as long as $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$,

(e) (*A lemma for the Riemann mapping theorem*) Let $\Omega \subseteq \mathbb{C}$ is an open set and let K be the complement of $S(\Omega)$ in the Riemann sphere. Note that $(0,0,1) \in K$ for any such Ω. Show that if K is not connected, then Ω^c (the complement of $\Omega \in \mathbb{C}$) has at least one bounded component. Since Ω^c is closed, this bounded component must be compact.

(f) Let Ω be the strip defined by $z = x + iy$ with $|y| < 1$, Show that the complement of $S(\Omega)$ in the Riemann sphere is connected.

(g) Define the linear fractional transformation $f(z) = \frac{1}{z}$ and the image of the strip $\Omega' = f(\Omega)$. Describe Ω' and show that it is simply connected (by the definition that the complement of Ω' "in the Riemann sphere" is connected).