Complex Variables II
Assignment 10

1. The *support* of a function $f(x)$ is the closure of the set where $f \neq 0$. A function has *compact support* if the support is a compact set. More simply, there is an R so that $f(x) = 0$ for $|x| \geq R$. Show that it is impossible for f and its Fourier transform $\widehat{f}$ both to have compact support. *Hint.* If f has compact support then the Fourier integral defines an analytic function $\widehat{f}(\xi + i\eta)$ defined for all $\zeta = \xi + i\eta$. The integral defines a complex differentiable function of ζ.

2. Define $f(x)$ as the characteristic function (also called *indicator function* of $[-1,1]$, in various notations:
$$f(x) = \chi_{[-1,1]}(x) = \mathbf{1}_{[-1,1]}(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise .} \end{cases}$$
   (a) Calculate $\widehat{f}(\xi)$.
   (b) Show by explicit calculation that the Fourier inversion formula holds in the sense that if $x \neq \pm 1$
$$f(x) = \lim_{R\to\infty} \int_{|\xi|\leq R} e^{2\pi i \xi x} \widehat{f}(\xi)\, d\xi \tag{1}$$
   (c) Show by explicit calculation that the integral on the right of (1) is equal to $\frac{1}{2}$ if $x = \pm 1$. *Discussion.* Let f have a simple discontinuity at a point x_0 with left and right limits defined:[1] $f_- = f(x_0 - 0)$ and $f_+ = f(x_0 + 0)$. Then the inverse Fourier transform of $\widehat{f}$ is equal to $\frac{1}{2}(f_- + f_+)$ at the discontinuity. You are not being asked to prove this general fact, only to check it for the characteristic function. *Discussion.* These calculations illustrate the theory of Exercise 1. We see that f has compact support and $\widehat{f}$ is analytic.

3. The *theta function* (more properly, the *Jacobi* theta function) is
$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{2\pi i z n} . \tag{2}$$
This is defined for all $z \in \mathbb{C}$ and all complex τ in the upper half plane ($\mathrm{Im}(\tau) > 0$). The theta function is used in analytic number theory, for example, in proving the Riemann functional equation for the Riemann zeta function. It also

[1] $f(x+0)$ is defined to be $\lim_{\epsilon\to 0, \epsilon>0} f(x+\epsilon)$.

(a) Show that $\theta(z,\tau)$ is an analytic function of z and τ if $\text{Im}(\tau) > 0$.

(b) Show that $\theta(z+1,\tau) = \theta(z,\tau)$.

(c) Show that $\theta(z+\tau) = e^{-i\pi\tau - 2\pi iz}\theta(z,\tau)$. Because of this formula, people say that τ is a *quasi-period* of the theta function.

(d) Let γ be a simple counter-clockwise contour around the corners of a "period cell". That means that γ is piecewise linear as it takes the path $a \to a+1 \to a+1+\tau \to a+\tau \to a$. Choose any a so that $\theta(z,\tau) \neq 0$ for $z \in \gamma$. Compute the winding number of θ (as a function of z) around γ and show that there is a unique z_0 in a period cell with $\theta(z_0,\tau) = 0$. *Hint.* part (c).

(e) Find a function $f(x,z,\tau)$ so that $\theta(z,\tau) = \sum_n f(n,z,\tau)$. *Warning.* Here, x is a new variable, not the real part of z. Apply the Poisson summation formula (Assignment 9, Exercise 5) to find a formula for $\theta(z,-1/\tau)$. This is the *Jacobi inversion formula.*

4. In class we said that a *Riemann surface* is a metric space R so that every $p \in R$ has a neighborhood (connected open set containing p) U_p and a continuous one to one function ϕ_p: $U_p \to V_p \subseteq \mathbb{C}$, called a *coordinate chart* at p. The inverse map ϕ_p^{-1}: $V_p \to U_p$ must be continuous. If $q \in U_p$, then $U_q = U_p$ and $\phi_q = \phi_p$ is allowed. We say that $z = \phi_p(r)$ is a *local variable* defined for $r \in U_p$. If q is another point in R and $r \in U_q$, then $w = \phi_q(r)$ is a possibly different local coordinate at r. The map $z \to w$, where it is defined, must be analytic in the usual sense. More precisely, suppose $U_{pq} = U_p \cap U_q \neq \emptyset$. Define the image of U_{pq} under ϕ_p as $V_{pq} = \phi_p(U_{pq}) \subset V_p \subset \mathbb{C}$. (Warning: clumsy notation ahead) Define $V'_{pq} = \phi_q(U_{pq}) \subset V_q \subset \mathbb{C}$. The *overlap map* that expresses the change of variable from z to w goes from V_{pq} (where z is defined) to V'_{pq} (where w is defined) via R:

$$\psi_{pq}(z) = w \;\text{ if } w = \phi_q(r) \;, \;\; \text{where } r = \phi_p^{-1}(z) \;.$$

   That is, if $r \in U_p \cap U_q$, then z and w are different labels for r. The hypothesis is that all such overlap maps ψ_{pq} are one to one and analytic.

   Define the Riemann sphere S as follows (see if you can spot the ignored details). As a set, S is consists of all "finite points" $z \in \mathbb{C}$ plus the "point at infinity" denoted by ∞. Suppose there are open sets $U_0 \subset S$ and $U_\infty \subset S$ defined by

$$U_0 = \{z \in \mathbb{C} \text{ with } |z| < 2\}$$
$$U_\infty = \left\{z \in \mathbb{C} \text{ with } |z| > \tfrac{1}{2}\right\} \cup \{\infty\} \;.$$

   The corresponding coordinate maps defined $\phi_0(z) = z$, $\phi_\infty(z) = \frac{1}{z}$, and $\phi_\infty(\infty) = 0$.

5. A map between Riemann surfaces is analytic if it is analytic on each pair of coordinate charts. More precisely, suppose $f: R_1 \to R_2$ is a

map between Riemann surfaces. Suppose $p \in R_1$ and $f(p) = q \in R_2$. Then f "induces" a map from $V_p \subset \mathbb{C}$ to $V_q \subseteq \mathbb{C}$ (not necessarily onto) $f_{pq}(z) = \phi_q(f(\phi_p^{-1}))$. You could express this by saying that $f_{pq}(z) = w$ means that $\phi_p(r) = z$, $s = f(r)$, and $w = \phi_q(s)$. This is also written with arrows as $z \rightsquigarrow r = \phi^{-1}(z) \rightsquigarrow s = f(r) \rightsquigarrow w = \phi_q(s)$. Every such map f_{pq} is supposed to be analytic. Riemann surfaces R_1 and R_2 are *isomorphic* if there is a pair of onto analytic mappings $f : R_1 \to R_2$ and $g : R_2 \to R_2$ that are inverses of each other. For example, the Riemann mapping theorem says that any two simply connected open subsets of $\mathbb{C}$ are isomorphic (unless one of them is $\mathbb{C}$ itself).

A *lattice* in $\mathbb{C}$ is the set of all integer linear combinations of two lattice *basis vectors*. If τ_1 and τ_2 are linearly independent complex numbers, the lattice they generate is

$$\Lambda = \{n\tau_1 + m\tau_2 \text{ with } n \text{ and } m \text{ integers }\} \ .$$

The *complex torus* is the quotient $T = \mathbb{C}/\Lambda$.

(a) Show that T is a Riemann surface in the natural way in which the coordinate charts ϕ have inverses ϕ^{-1} that take $z \in \mathbb{C}$ to the coset $[z] = \{z + n\tau_1 + m\tau_2\}$. For any $[z_0] \in T$, there is a neighborhood of $z_0 \in \mathbb{C}$ in which this ϕ is one to one. Check that the overlap maps are analytic. *Warning.* This is almost obvious and yet might be a challenge to express in mathematically correct notation.

(b) An *automorphism* is an isomorphism from an object to itself. For example, $f(x) = x + x^3$ is an automorphism from $\mathbb{R}$ to $\mathbb{R}$. Show that any complex torus has automorphisms "translation by $a \in \mathbb{C}$" that take $[z]$ to $[z + a]$. Show that a and b induce the same translation automorphism if and only if $a - b \in \Lambda$.

(c) Show that if $f : T_1 \to T_2$ is an isomorphism between complex tori, then f *lifts* to a map $F : \mathbb{C} \to \mathbb{C}$ so that if $F(z) = w$ then $f([z]) = [w]$. *Warning.* Again, this may be obvious, but it may take some cleverness to create a proof. You might try to work by induction, defining F on more and more periodic images of the fundamental domain $D = \{x\tau_1 + y\tau_2 \text{ with } 0 \leq x < 1 \text{ and } 0 \leq y < 1\}$.

(d) Use Liouville's theorem and the bound $|F(z)| \leq C\,|z| + r$ for some positive numbers C and r to show that F is *affine*, which means that there complex a and b with $F(z) = az + b$. Conclude that any automorphism between T_1 and T_2 lifts to an affine automorphism of $\mathbb{C}$.

(e) Show that the complex torus $\mathbb{C}/\Lambda$ where Λ is generated by 1 and i (that means 1 and i are basis vectors for Λ) is identical to the one with generators 1 and $1 + i$.

(f) Show that there is no isomorphism between complex tori T_1 with basis vectors 1 and i and the one with basis vectors 1 and $\frac{1}{2} + i$.

Discussion. We will talk more about automorphisms of complex tori in class. The new thing involved is the group $SL2(\mathbb{Z})$, which is the set of matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (2×2 gives the 2 in $SL2$) with integer entries (hence the $\mathbb{Z}$) and $\det(M) = ad - bc = 1$ (which is the "S", for "special").