Assignment 4.

Given November 5, due November 19.

Objective: Basics of stochastic differential equations.

(1) The solution of the model stochastic differential equation

$$dX = XdZ \quad , \qquad X(0) = 1 \tag{1}$$

is given by

$$X(t) = \exp\left(Z(t) - t/2\right) \quad . \tag{2}$$

Here $Z(t)$ is a standard Brownian motion with $Z(0) = 0$. From (2) we get the formula for the density function for $X(t)$:

$$u(x,t) = \frac{1}{\sqrt{2\pi t}}\frac{1}{x}\exp\left(\frac{-\left(\log(x) + t/2\right)^2}{2t}\right) \quad . \tag{3}$$

(Check this yourself.) The forward Euler approximation gives $X_k \approx X(t_k)$ whose density function is $u(x, t_k, \Delta t)$. Use the histogram method to compute $u(x, t, \Delta t)$ and then compare it to the exact answer, $u(x,t)$ for $t = 1$ and Δt small but not too small; compare the curves. If the bins are too small, this will be impossible, it will require too large a sample size. Use values Δt, $\Delta t/2$, $\Delta t/4$, and so on to verify that $u(x, t, \Delta t)$ is a first order accurate approximation to $u(x,t)$, as curves.

(2) Redo the above with the complication that X is absorbed if $X(t) \geq 2$ for any $t \leq 1$. This time, do not use a formula for the exact density, $u(x,t)$, when you determine the accuracy of $u(x, t, \Delta t)$. In the forward Euler approximation you implement the absorption by absorbing a walker if $X_k \geq 2$ for any k with $t_k \leq 1$.

(3) Now turn (1) into a (simplified) stochastic volatility model by writing

$$dX = \sigma(t) dZ_1(t) \quad , \qquad X(0) = 1 \ , \tag{4}$$

$$d\sigma(t) = \gamma\sqrt{\sigma} dZ_2(t) \quad , \qquad \sigma(0) = 1 \ . \tag{5}$$

Here $Z_1(t)$ and $Z_2(t)$ are independent standard Brownian motions. For $\gamma = 0$, the X process is as before. Give Monte Carlo estimates of

$$F(\gamma, T, K) = \mathbf{E}\left[\left(X(T) - K\right)_+\right] \quad ,$$

for parameter values in the ranges: $.1 \leq \gamma \leq 5$, $.1 \leq T \leq 5$, and $.2 \leq K \leq 1$.[1]

[1]Please do not hand in hundreds or even tens of plots representing every possible combination of parameters.

Use two variance reduction strategies, antithetic variates and control variates. For the control variate case, the exactly solvable problem is the case $\gamma = 0$. The solution for this case is given by the Black–Scholes formula with $r = 0$ and $\sigma = 1$. Control variates work well when the approximation is good. Antithetic variates work well when the function being "expected" is nearly linear as a function of Z over the range of variation of Z. For which combinations of parameters do you expect these conditions to hold? Comment on how well the numerical experiments agree with your expectations.