Assignment 1.

Given September 24, due October 7.

Objective: To practice the formal methods of numerical analysis.

(1) Find the coefficients, a, b, and c, so that

$$\partial_x^2 f(x) = \frac{1}{\Delta x^2}\left[a(f(x+2\Delta x)+f(x-2\Delta x)) + b(f(x+\Delta x)+f(x-\Delta x)) + cf(x)\right] + O(\Delta x^4) \quad .$$

Are these coefficients uniquely determined?

(2) Find the coefficients, a, b, and c, so that

$$\partial_x^4 f(x) = \frac{1}{\Delta x^4}\left[a(f(x+2\Delta x)+f(x-2\Delta x)) + b(f(x+\Delta x)+f(x-\Delta x)) + cf(x)\right] + O(\Delta x^2) \quad .$$

Are these coefficients uniquely determined?

(3) Show that if $u(x,t)$ satisfies the heat equation

$$\partial_t u = \frac{1}{2}\partial_x^2 u \tag{1}$$

then

$$\partial_t^2 u = \frac{1}{4}\partial_x^4 u \quad .$$

Use this to show that if u satisfies (1) then

$$u(x, t+\Delta t) = u(x,t) + \frac{1}{2}\Delta t \partial_x^2 u(x,t) + \frac{1}{8}\Delta t \partial_x^4 u(x,t) \quad . \tag{2}$$

(4) Show that if the terms on the right of (2) are discretized using the formulas from problem 1 and problem 2, then a fourth order accurate explicit scheme for the heat equation arises if $\Delta t = \lambda \Delta x^2$.

(5) Use whatever methods you can (graphical or analytical) to determine for which values of λ the scheme from problem 4 is stable.

(6) Use one of the computations from the first assignment to check that this fourth method is indeed fourth order accurate, if the stability limit is respected.