Honors Algebra II, Courant Institute, Spring 2022
http://www.math.nyu.edu/faculty/goodman/teaching/HonorsAlgebraII2022/HonorsAlgebraII.html
Check the class forum for corrections and hints

Assignment 8

Due: Thursday, April 14.

From the textbook exercises for Chapter 13:

- 4.1
- 5.2 (this shows that $\mathbb{Z}[i\sqrt{3}]$ is not the ring of algebraic integers in $\mathbb{Q}[i\sqrt{3}]$.
- 6.3
- 7.1

Multiplicative property of determinants:

$$\det(BA) = \det(B)\det(A)\ . \tag{1}$$

There are several proof strategies. One works directly from formulas for matrix multiplication,l the determinant formula (1) from Assignment 6, and properties of the permutation group. Another shows that there is a unique (up to a constant multiplier) function with the properties of determinant: linear in each row and column, skew symmetric under row or column exchange. Both sides of (1) have these properties for A factor (or the B factor), so they must be equal. The approach in this assignment uses the practical fact that the determinant may be found from the upper triangular form of of the matrix. Here is one version of that argument:

1. Show that if (1) holds in $\mathbb{C}$, then it holds in any ring. *Hint.* The "universal polynomial" method from Assignment 7 applies.
2. Show that (1) holds for any two complex matrices if it holds for all complex matrices with $|a_{ij} - \delta_{ij}| \leq \epsilon$ and $|b_{ij} - \delta_{ij}| \leq \epsilon$. Here, δ_{ij} is the Kronecker delta symbol.
3. The "elementary row operation matrix" $E_{jk}(m)$ is defined by the property that $E_{jk}(m)A$ adds $m \cdot$ (row j) to row k of A. The following properties of elementary matrices are/should be familiar from linear algebra:
   (a) Identify the entries of $E_{jk}(m)$.
   (b) Express $E_{jk}^{-1}(m)$ in the form of an elementary matrix. *Hint.* There is a very simple abstract proof that does not use the specifics of part (a).
   (c) Describe $BE_{jk}(m)$ as an operation on the columns of B.

(d) Show that $\det(E_{jk}(m)\,A) = \det(A)$ and $\det(B\,E_{jk}(m)) = \det(B)$. *Warning.* Do this directly from properties of the determinant (multilinear, skew under row or column operations) rather than by (1) because this is a step in the proof of (1).

4. A matrix U is *upper triangular* if its entries satisfy $u_{jk} = 0$ for $j > k$. The entries $j > k$ are the ones "below" the main diagonal. Show that if the condition of Exercise 2 is satisfied, then (with some abuse of notation) there are elementary matrices $E_i = E_{j_i,j_i}(m_i)$ so that $E_1 E_2 \cdots E_m A = U$, where U is upper triangular. *Remarks.*

   (a) The hypotheses from Exercise 2 imply that "pivoting" is not required. This should feature in your solution. The conclusion of Exercise 4 is false for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   (b) Repeated application of Exercise 3(d) shows that $\det(A) = \det(U)$.

   (c) $\det(BA) = \det(\widetilde{B}U)$ where $\widetilde{B} = BE_m^{-1} \cdots E_1^{-1}$, and $\det(B) = \det(\widetilde{B})$, and $\widetilde{B}$ satisfies the inequalities of Exercise 2, with a larger ϵ (why?).

   (d) Exercise 4 may be applied to $\widetilde{B}$ to show that there is a sequence of elementary matrices F_i so that $F_1 \cdots F_m \widetilde{B} = V$ is upper triangular and (this is the main trick of this assignment) that $\det(VU) = \det(\widetilde{B}U)$.

5. Show that if U is upper triangular, then its determinant is the product of its diagonal entries:
$$\det(U) = u_{11} \cdots u_{nn}\ .$$

6. Show that if V and U are upper triangular, then VU is upper triangular and the diagonal entries of VU are products of diagonals of V and U
$$(VU)_{ii} = v_{ii} u_{ii}\ .$$
Use this to show that (1) holds for upper triangular matrices.

7. Use arguments from Assignment 7 to show that (1) holds in any ring.