Honors Algebra II, Courant Institute, Spring 2022
http://www.math.nyu.edu/faculty/goodman/teaching/HonorsAlgebraII2022/HonorsAlgebraII.html
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Assignment 6

Due: Thursday, March 31.

This assignment is about determinants. This material "should" be familiar from linear algebra, but how much is familiar will depend on which linear algebra class you took. As usual, each exercise makes use of the exercises that come before.

1. Determinants are closely related to permutations. Let $I_n = \{1, 2, \cdots, n\}$ be the set of indices from 1 to n. A *permutation*, π, is a map $\pi\colon I_n \to I_n$ that is onto. Show that if $\pi\colon I_n \to I_n$ is onto, then it must be into, one to one, and invertible.

2. Show that the composition of onto maps is onto (for any spaces) and therefore that the set onto maps $I_n \to I_n$ forms a group. This is the *permutation group*, also called the *symmetric group*, and is denoted S_n.

3. The *transposition* $\tau_{ij} \in S_n$, is the permutation that interchanges i with j without moving any other index. That is $\tau_{ij}(k) = k$ if $k \neq i$ and $k \neq j$, while $\tau_{ij}(i) = j$ and $\tau_{ij}(j) = i$. Show that transpositions generate the permutation group. As a reminder, this means that every $\pi \in S_n$ may be written as a *word* consisting of transpositions (a product of transpositions). It also means that any subgroup of S_n that contains every transposition is S_n.

4. Consider a permutation that consists of a chain of transpositions:

$$\pi = \tau_{i_1,j_1} \cdot \tau_{i_1,j_2} \cdot \; \cdots \; \cdot \tau_{i_M,j_M} \;.$$

This means that you can apply the function $\pi : I_n \to I_n$ by first interchanging i_M with j_M, then doing a chain of transpositions ending with interchanging i_1 with j_1. There can be more than one way to express π as a product of transpositions, and different products for the same π may have different lengths. Show that if π is expressed by a chain of length M and another one of length L, with L possibly not equal to M, then $(-1)^L = (-1)^M$ (both L and M are even or both are odd). Conclude that the signature of a permutation, $\text{sign}(\pi) = (-1)^M$, is well defined and satisfies $\text{sign}(\pi)\text{sign}(\sigma) = \text{sign}(\pi\sigma)$ (for permutations π and σ). *Hint.* You can transform one chain into another using the following facts about transpositions: (i) $\tau_{ij}^2 = \text{Id}$, (ii) $\tau_{ij}\tau_{kl} = \tau_{kl}\tau_{ij}$ if the pair (i, j) is disjoint from the pair (k, l) ("disjoint" transpositions commute), (iii) $\tau_{ij}\tau_{jk}\tau_{ij} = \tau_{ik}$ (the *commutator* of τ_{12} and τ_{23} is τ_{13}, three transpositions are replaced by one transposition).

5. Suppose A is an $n \times n$ matrix with entries $a_{ij} \in R$, with R being a ring. The *determinant* is the sum

$$\det(A) = \sum_{\pi \in S_n} \left[\operatorname{sign}(\pi) \prod_{j=1}^{n} a_{j,\pi(j)} \right] . \tag{1}$$

$\boxed{\texttt{det}}$

Show that $\operatorname{sign}(\pi) = \operatorname{sign}(\pi^{-1})$ for any $\pi \in S_n$. Show that $\det(A) = \det(A^t)$.

6. The *complementary matrix* for any (i, j) is the $(n-1) \times (n-1)$ matrix with row i and column j removed. In this exercise, this will be expressed as $\overline{A}_{ij}$. The *cofactor* determinant is the determinant of the complementary matrix:

$$\overline{a}_{ij} = \det(\overline{A}_{ij}) \ .$$

A matrix B has rank 1 if its entries may be expressed as $b_{jk} = u_j v_k$. This is equivalent to the usual definition of rank of a matrix as the dimension of the subspace spanned by its columns (recall the theorem that "column rank" and "row rank" are equal). This may be written $B = uv^t$, where u and v are the column vectors with entries u_j and v_k. Show that $\det(A + tB) = \det(A) + \beta t$, with $\beta = u^t M v$ and identify the $n \times n$ matrix M. *Hint.* This can be done "directly" by the determinant definition (1). But it may be less confusing and involve less writing to do it more abstractly. The determinant formula makes it clear (how?) that $\det(A + uv^t$ is *bilinear* in u and v, meaning that is is a linear in each entry u_j and v_k and zero on all products $u_i u_j$ and $v_k v_l$. You can identify entry M_{jk} by taking $u = e_j$ and $v = e_k$ (the "standard basis").

7. The *cofactor expansion* of the determinant for the first row is

$$\begin{aligned} \det(A) &= a_{11}\overline{a}_{11} - a_{12}\overline{a}_{12} + a_{13}\overline{a}_{13} - \cdots \qquad (2) \\ &= \sum_{i=1}^{n} (-1)^{i-1} a_{1i}\overline{a}_{1i} \ . \end{aligned}$$

$\boxed{\texttt{cof}}$

Prove this formula. Prove the corresponding formula for cofactor expansion along the first column, or any row or column. *Hint.* For (2), collect all the terms in the sum (1) where the product includes a_{11}, or a_{21}, etc.

8. Use the cofactor expansion to calculate the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$, for the matrix in class that represents the action of multiplication by $\alpha = \sqrt{2} + \sqrt{3}$ in the generators $g_1 = 1$, $g_2 = \sqrt{2}$, $g_3 = \sqrt{3}$, $g_4 = \sqrt{6}$. Here A is a 4×4 integer matrix. The cofactor expansion gives the determinant in terms of determinants of 3×3 matrices, and then in terms of determinants of 2×2 matrices. If you do it all correctly, the polynomial satisfies $f(\alpha) = 0$.

9. The *cofactor matrix* will be called B (though it might be called $\overline{A}$ or A' in books). Its entries, b_{ij}, are cofactor determinants, with signs, and transposed:

$$C = \begin{pmatrix} \overline{a}_{11} & -\overline{a}_{21} & \overline{a}_{31} & \cdots \\ -\overline{a}_{12} & \overline{a}_{21} & -\overline{a}_{31} & \cdots \\ \vdots & & & \end{pmatrix}$$

$$c_{ij} = (-1)^{i+j-1}\,\overline{a}_{ji} \; .$$

Show that $AC = \det(A)I$.

10. Let a_i be the $n-$component column vector that is column i of the matrix A. You can think of A as a collection of these columns:

$$A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \quad , \quad a_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix} \; .$$

The determinant is a function of the columns

$$\det(A) = D(a_1, \cdots, a_n) \; .$$

Show that D is linear in its first argument, or any of the others. That is:

$$D(a_1 + mb_1, a_2, \cdots, a_n) = D(a_1, a_2, \cdots, a_n) + mD(b_1, a_2, \cdots, a_n) \; .$$

11. Show that transposing columns reverses the sign of D. For example,

$$D(a_1, a_2, a_3, \cdots) = -D(a_2, a_1, a_3, \cdots) \; .$$

Continued next assignment. Some properties of determinants are missing, so far. In particular, the multiplicative property $\det(AB) = \det(A)\det(B)$. This can be verified directly from (1) using matrix multiplication and manipulations involving S_n. Assignment 7 will have a different approach that one might be more likely to remember.