Assignment 4

Due Thursday, March 3.

1. Give the proof that $\mathbb{Z}[X]/(X^2-1)$ is not isomorphic to $\mathbb{Z}[X]/(2X-1)$. *Note*. This was on the quiz and was discussed in detail in class. It is in the assignment to make sure you can write the proof correctly.

2. The *wikigroup* (not its real name), G, consists of 2×2 matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and determinant 1. Show that this group is generated by matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;,\quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;.$$

   Hint. One proof uses reasoning like Euclid's gcd algorithm. You can simplify M in a sequence of steps using T, R, T^{-1}, and R^{-1}, until it is simplified to $I \in G$. The hypothesis $\det(M) = 1$ has number-theoretic consequences. *Background*:

   (a) Let $\sigma_1, \cdots, \sigma_d$ be distinct *symbols*. A *word*, V, of length n is a sequence of n symbols: $V = S_1 S_2 \cdots S_n$, where each "letter" is one of the symbols: $S_k = \sigma_{j_k}$. Let T and R be elements of G and consider the four symbols $\sigma_1 = T$, $\sigma_2 = R$, $\sigma_3 = T^{-1}$, and $\sigma_4 = R^{-1}$. Every word made from these symbols corresponds to some element of G, for example

$$\sigma_1\sigma_2 = TR = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \;,\;\; \sigma_1\sigma_3\sigma_2 = TT^{-1}R = R \;.$$

   The second example shows that there can by more than one way to "spell" an element of G. The correspondence between words and elements of G is not $1-1$. The elements T and R *generate* G if every $M \in G$ can be "spelled" by a word using these four symbols.

   (b) The elements T and R *generate* G if there is no proper subgroup $H \subset G$ with $T \in H$ and $R \in H$.

   (c) The definitions (a) and (b) are equivalent. The element corresponding to any word must be in H, as H is closed under multiplication by T, R, T^{-1} and R^{-1}. The elements of G corresponding to words form a subgroup (closed under inverse and multiplication), which would be proper if words did not describe every element of G.

Textbook exercises from Chapter 12
1.2, 1.3, 1.5, 2.1, 2.2, 2.6(a), 2.9