Quiz, February 25.

Quiz Instructions and information

- You must do the quiz while on zoom with your camera on at all times. Upload your answers at the end of the quiz.
- The quiz will be closed book, closed notes, etc. You may not use any resources during the quiz, except …
- You may prepare and use a "cheat sheet", which is one US standard size ($8\frac{1}{2}'' \times 11''$) piece of paper, front and back. Please upload the cheat sheet with your quiz answers, if you use one.
- Please write as clearly and neatly as possible in a quiz situation. If you scan or photograph a handwritten paper (the most common mode), please do that as well as possible in the quiz setting.
- You will be graded on clarity as well as mathematical correctness. You don't have to use full sentences in each case, but what you write should be grammatical and use mathematical terms and notation correctly. You may use scratch paper that you don't hand in to organize your thoughts. Reasoning is as important as the answer in a theory class like abstract algebra.
- You will get 25% credit for any question or question part that you leave blank. You may lose points for a wrong answer, even if you also give a correct answer. Cross out anything you think is wrong.
- If you have a question during the quiz, please communicate with me on chat in the zoom session. Make sure the chat goes only to me and not the everyone in the session.

Questions

1. Let $\mathbb{F}_p$ be the finite field with p elements. A positive prime p is a *characteristic* of a ring or a field if $x + \cdots + x = 0$ (p summands on the left) for all x in the ring or field.
   (a) Show that $\mathbb{F}_p$ has characteristic p.
   (b) Show that the polynomial ring $R = \mathbb{F}_p[x]$ has characteristic p.
   (c) Show that if $\mathbb{K}$ is a field of characteristic n, then n is prime.
   (d) Is it possible that a non-trivial ring (a ring with more than one element) has more than one prime characteristic? (Give an example or prove that it is not possible.)

2. Find a prime factorization of $5i$ in the Gaussian integers $\mathbb{Z}[i]$.

3. Define $\alpha = 5^{\frac{1}{3}}$, which is the positive real number with $\alpha^3 = 5$. Define $\beta = e^{2\pi i/3}\alpha$. Define $\mathbb{F} = \mathbb{Q}[\beta] \subset \mathbb{C}$.

   (a) Show that $\mathbb{F}$ is a vector space over $\mathbb{Q}$ of dimension 3.

   (b) Show that $\mathbb{F}$ is a field.

   (c) Construct a field isomorphism $\sigma\colon \mathbb{F} \to \mathbb{Q}[\alpha]$.

4. An ideal $I \subset R$ is a *prime ideal* if for any $f, g \in R$, if $fg \in I$ then $f \in I$ or $g \in I$.

   (a) Show that if $p \in R$ is a prime element then $I = (p)$ is a prime ideal. Here (p) is the principal ideal generated by p.

   (b) Let $R = \mathbb{C}[x, y]$ (polynomials in two variables with complex numbers as coefficients). Show that $h(x, y) = x$ is prime element of R.

   (c) Define $I_x = \{f \in R \mid f(0, y) = 0 \text{ for all } y \in \mathbb{C}\}$. Show that I_x is a prime ideal. *Hint*: Let $(fg)(x, y)$ be the product of polynomials f and g. If $(fg)(0, y) = 0$ for all y, then either $f(0, y) = 0$ or $g(0, y) = 0$ for infinitely many values of y.

   (d) Show that $(x) \subset I_x$. Here $(x) \subset R$ is the principal ideal generated by the polynomial $h \in R$ with $h(x, y) = x$.

   (e) Show that $(x) = I_x$. *Hint.* If $f \in I_x$, what can you say about all the monomials in f?

   (f) Define $J = \{f \in R \mid f(0, 0) = 0\}$. Show that J is a prime ideal but not a principal ideal.