Assignment 7, due March 23 (before class starts).

Instructions

- Do not hand in a rough draft. Copy or type answers neatly and clearly. Points may be deducted for writing that is sloppy, has excessive cross-outs, or is hard to read.
- State facts precisely in clear language or notation. Put assertions in logical order. State clearly what the hypotheses and conclusions. Put the steps of an argument in logical order, including definitions. Points may be deducted for an incorrectly stated argument even if you seen to understand it. Clear mathematical exposition is an important goal for the class.
- Learn the Greek letters used in math. Learn their mathematical names and write them clearly.

Suggested exercise, not to hand in

1. Take $f \in \mathbb{Q}[t]$ to be $f(t) = (t^2+1)(t^2-2)$. Let $\mathbb{K}/\mathbb{Q}$ be a splitting field of f. For concreteness, you can take $\mathbb{K} \subset \mathbb{C}$, but this is not necessary. Carry out the construction described in Section 16.3 starting with the element $\beta_1 = i + \sqrt{2} \in \mathbb{K}$. What is the orbit of β_1 under the action of the permutation group on the roots of f? What is the polynomial $h(t) = \prod(t-\beta_j)$? If it is reducible, find the irreducible whose root is β_1.

Assigned Exercises, to hand in

1. Exercise 2.2 from Chapter 16
2. Exercise 2.7 from Chapter 16 *Discussion.* A_n is the *alternating group*, which is the group of *even* permutations of n objects. This is a subgroup of S_n (the full permutation group) of order 2. Every permutation may be achieved using a sequence of binary exchanges $j \leftrightarrow k$. A permutation is even if it is achieved using an even number of exchanges. There is more than one way to achieve a given permutation through exchanges, but it is a theorem that they have the same even/odd *parity*. See Exercise 7 for more on this.
3. Exercise 3.1 from Chapter 16 *Hint.* it's easy to divide $n!$.
4. Exercise 3.2 from Chapter 16
5. Exercise 3.3 from Chapter 16 *Extra question.* What (false) "theorem" is this a counter-example for? Is this possible over $\mathbb{Q}$?

6. Suppose $f \in \mathbb{Z}[t]$ is irreducible over $\mathbb{Z}$, $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Let α_j be the distinct roots of f in an extension field of $\mathbb{Q}$. Find a formula for

$$S = \sum_{j=1}^{n} \alpha_j^2 \, .$$

7. The *Vandermonde* matrix with n variables is the $n \times n$ matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

(a) Verify the Vandermonde determinant formula

$$\det(V)(x_1, \cdots, x_n) = p(x_1, \cdots, x_n) = \prod_{i<j} (x_j - x_i) \, . \qquad (1)$$ V

Hint. One way to do this is to show that each $x_j - x_i$ is a factor and that these are the only factors. Another way is do see it directly using Gaussian elimination. It could start be eliminating ones in the first column then factoring out $x_j - x_1$, as:

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & \cdots & x_2^{n-1} - x_1^{n-1} \\ \vdots & \vdots & & & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_1^2 & \cdots & x_n^{n-1} - x_1^{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 0 & (x_2 - x_1) & (x_2 - x_1)(x_2 + x_1) & \cdots & (x_2 - x_1)(\cdots) \\ \vdots & \vdots & & & \vdots \\ 0 & (x_n - x_1) & (x_n - x_1)(x_n + x_1) & \cdots & (x_n - x_1)(\cdots) \end{pmatrix}$$

$$\det(V) = (x_2 - x_1) \cdots (x_n - x_1) \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 0 & 1 & (x_2 + x_1) & \cdots & (\cdots) \\ \vdots & \vdots & & & \vdots \\ 0 & 1 & (x_n + x_1) & \cdots & (\cdots) \end{pmatrix} .$$

The last line uses the fact that multiplying a row of a matrix by a constant multiplies the determinant by the same constant. Then you have to figure out why the determinant on the right is the Vandermonde determinant of $x_2, \cdots, x_n$. The sub-matrix is the $(n-1) \times (n-1)$ matrix

$$\widetilde{V} = \begin{pmatrix} 1 & (x_2 + x_1) & \cdots & (\cdots) \\ 1 & (x_3 + x_1) & \cdots & (\cdots) \\ \vdots & & & \vdots \\ 1 & (x_n + x_1) & \cdots & (\cdots) \end{pmatrix} .$$

Note that the entries in the last column are $(\cdots) = x_j^{n-2} + x_j^{n-3}x_1 + \cdots + x_1^{n-2}$. The intermediate columns have a similar structure with a power lower than $n-2$. Do column operations that subtracts x_1 times the first column from the second column, x_1^2 times the first column from the third, etc., and you get

$$\det\widetilde{V} = \det \begin{pmatrix} 1 & x_2 & x_2^2 + x_2x_1 & \cdots \\ 1 & x_3 & x_3^2 + x_3x_1 & \cdots \\ \vdots & & & \vdots \\ 1 & x_n & x_n^2 + x_nx_1 & \cdots \end{pmatrix}.$$

Another column operation removes the terms x_jx_1 from the third column. This approach may seem longer, but it gives a clearer idea what is "going on".

(b) Suppose permutations $\sigma \in S_n$ act on polynomials in n variables by permuting the arguments. Find the orbit of the Vandermonde determinant $p(x_1, \cdots, x_n)$ under this action.

(c) Consider $t^3 - 2 \in \mathbb{Q}[t]$ with roots $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{C}$. Calculate the Vandermonde determinant $p(\alpha_1, \alpha_2, \alpha_3$ to show that it is not in the ground field $\mathbb{Q}$ but is in a quadratic extension. If you are not careful this calculation could take quite a while. Try to avoid that using tricks. Square this to get the discriminant D (defined in Section 6.2). I believe the answer is $D = -108$, but I could be wrong.

(d) Prove or disprove the following conjecture. Let $\mathbb{F}$ be a field and f an irreducible polynomial over $\mathbb{F}$. Let $\mathbb{K}$ be a field in which f splits. Let $p(x_1, \cdots, x_n)$ be the Vandermonde determinant of the roots of f. Then $u = p(\alpha_1, \ldots, \alpha_n) \notin \mathbb{F}$. Does it matter whether $\mathbb{F}$ has characteristic zero?