Assignment 4, due March 2 (before class starts).

Instructions

- Do not hand in a rough draft. Copy or type answers neatly and clearly. Points may be deducted for writing that is sloppy, has excessive cross-outs, or is hard to read.
- State facts precisely in clear language or notation. Put assertions in logical order. State clearly what the hypotheses and conclusions. Put the steps of an argument in logical order, including definitions. Points may be deducted for an incorrectly stated argument even if you seen to understand it. Clear mathematical exposition is an important goal for the class.
- Learn the Greek letters used in math. Learn their mathematical names and write them clearly.

Assigned Exercises, to hand in

1. Exercise 4.1 from Chapter 15
2. Exercise 4.3 from Chapter 15
3. Exercise 5.2 from Chapter 15
4. Consider the field $\mathbb{K} = \mathbb{Q}[\alpha, \omega] \subset \mathbb{C}$, where $\alpha = 2^{\frac{1}{3}} \in \mathbb{R}$ and $\omega = e^{2\pi i/3}$. This exercise illustrates methods from linear algebra being combined with facts about fields.
   (a) Show that the following vectors are a basis of $\mathbb{K}$ as a vector space over $\mathbb{Q}$:
   $$u_1 = 1\ ,\ \ u_2 = \alpha\ ,\ \ u_3 = \alpha^2\ ,\ \ u_4 = \omega\ ,\ \ u_5 = \omega\alpha\ ,\ \ u_6 = \omega\alpha^2\ .$$
   That is, suppose $f \in \mathbb{Q}[x, y]$ is any polynomial in two variables with rational coefficients. Show that there are rational numbers so that $f(\alpha, \omega) = \sum_{j=1}^{6} r_j u_j$.
   (b) Show that $\mathbb{K}$ is a field. We already know it's a ring, so you need to show that if $x \neq 0 \in \mathbb{K}$, then there is a $y \in \mathbb{K}$ so that $xy = 1$. The argument we used before applies: For any x, define the map $\phi\colon \mathbb{K} \to \mathbb{K}$ defined by $\phi(z) = xz$. Show that this is a linear map over $\mathbb{Q}$ that has no kernel and therefore an isomorphism of $\mathbb{K}$ as a vector space over $\mathbb{Q}$. Then take $y = \phi^{-1}(1)$.
   (c) Show that ϕ from part (b) is not a field isomorphism.

(d) For any $x \in \mathbb{K}$, with $x = \sum_{j=1}^{6} r_j u_j$, we can organize the r_j into a column vector $r \in \mathbb{Q}^6$ with

$$r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} .$$

Find the column vectors corresponding to $x^0 = 1$, $x = \alpha + \omega$, and x^2, and x^3. These form the entries of a 4×6 matrix.

(e) Show by elementary row or column reduction (Gaussian elimination from linear algebra) that these four vectors are linearly independent over $\mathbb{Q}$. For example, if $r, s \in \mathbb{Q}^6$, and if $r_1 \neq 0$, then there is an l_{rs} so that the vector $\widetilde{s} = s - l_{rs} r$ has $\widetilde{s}_1 = 0$. Recall from linear algebra that you may have to "pivot" (re-order the rows and/or columns) or choose to "eliminate" numbers in a row that is not the top row.

(f) Show that $\mathbb{K} = \mathbb{Q}[x]$. You can do this by showing that the minimal non-zero polynomial $f \in \mathbb{Q}[t]$ with $f(x) = 0$ has degree 6. Part (e) implies that the degree is at least 4. Once you know that, field theory (multiplicity of degrees of extensions) implies that the degree cannot be 4 or 5 (why?).