Assignment 3, due February 23 (before class starts).

Instructions

- Do not hand in a rough draft. Copy or type answers neatly and clearly. Points may be deducted for writing that is sloppy, has excessive cross-outs, or is hard to read.
- State facts precisely in clear language or notation. Put assertions in logical order. State clearly what the hypotheses and conclusions. Put the steps of an argument in logical order, including definitions. Points may be deducted for an incorrectly stated argument even if you seen to understand it. Clear mathematical exposition is an important goal for the class.
- Learn the Greek letters used in math. Learn their mathematical names and write them clearly.

Assigned Exercises, to hand in

1. Exercise 3.1 from Chapter 15.
2. Exercise 3.2 from Chapter 15.
3. Exercise 3.3 from Chapter 15.
4. Let $\mathbb{Q} = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_n$ be a tower of field extensions. Suppose that $[\mathbb{F}_{k+1} : \mathbb{F}_k] = 3$ for each extension. Show that no element $\alpha \in \mathbb{F}_n$ satisfies $\alpha^2 = -1$.
5. [This exercise is related to the concept of *algebraic closure* of a field. We will do this more generally after we cover Section 15.6. Here is a special case of an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. An algebraic closure of field $\mathbb{F}$ is denoted $\overline{\mathbb{F}}$. This exercise defines algebraic closure and gives a construction of the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$.] A field $\mathbb{K}$ is *algebraically closed* if any polynomial with coefficients in $\mathbb{K}$ has a root in $\mathbb{K}$. If $\mathbb{F} \subset \mathbb{K}$ and $\mathbb{K}$ is algebraically closed, we say $\mathbb{K}$ is a *closing field* for $\mathbb{F}$. A closing field is an *algebraic closure* of $\mathbb{F}$ if no proper sub-field of $\mathbb{K}$ is a closing field for $\mathbb{F}$. It is a theorem that every field has an algebraic closure. We will not do the full proof in this class because it depends on *Zorn's lemma*, which takes a while to explain and appreciate. You may assume that theorem for part (a) of this exercise.
   (a) Consider the algebraic closure of the field of rational functions with coefficients in $\mathbb{C}$. This field is $\overline{\mathbb{C}(x)}$. Show that this is a closing field for $\mathbb{Q}$ but is not the algebraic closure of $\mathbb{Q}$.

(b) Suppose $\mathbb{F} \subset \mathbb{F}_1 \cdots \subset \mathbb{F}_n \subset \cdots$ is a countably infinite sequence of field extensions. Denote the union by $\mathbb{K}$:

$$\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{F}_n \ .$$

Show that $\mathbb{K}$ is a field and $\mathbb{F} \subset \mathbb{K}$ is a field extension.

(c) Suppose $\mathbb{K}$ is a closing field for $\mathbb{F}$. Show that $\mathbb{K}$ is an algebraic closure of $\mathbb{F}$ if and only if every $\alpha \in \mathbb{K}$ is algebraic over $\mathbb{F}$.

(d) (*uniqueness*) Show that if $\mathbb{K} \subset \mathbb{L}$ and $\mathbb{K}' \subset \mathbb{L}$ (same $\mathbb{L}$) and $\mathbb{K}$ and $\mathbb{K}'$ are algebraic closures of $\mathbb{F}$, then $\mathbb{K} = \mathbb{K}'$.

(e) Show that $\overline{\mathbb{Q}} \subset \mathbb{C}$ exists and is countable. *Hint.* We know $\mathbb{Q}$ is countable. Show that $\mathbb{Q}[x]$ is countable and therefore that there are countably many $\alpha \in \mathbb{C}$ that are algebraic over $\mathbb{Q}$.

(f) Suppose that $\mathbb{K} = \mathbb{F}(\alpha)$ and $\mathbb{K}' = \mathbb{F}(\beta)$, and that α and β have the same irreducible polynomial if $\mathbb{F}[x]$. Construct a map $\phi : \mathbb{K} \mapsto \mathbb{K}'$ that is a field isomorphism.

(g) Suppose $\phi\colon \mathbb{K} \mapsto \mathbb{K}'$ is a field isomorphism and that $\mathbb{L} = \mathbb{K}(\alpha)$ and $\mathbb{L}' = \mathbb{K}'(\beta)$. Let $f \in \mathbb{K}[x]$ be the irreducible monic polynomial for α and assume $f' = \phi(f)$ is the irreducible monic polynomial for β [If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\phi(f) = x^n + \phi(a_{n-1})x^{n-1} + \cdots + \phi(a_0)$.] Show that there is a unique field isomorphism $\psi\colon \mathbb{L} \mapsto \mathbb{L}'$ so that $\psi(a) = \phi(a)$ for $a \in \mathbb{K}$ and $\psi(\alpha) = \beta$. We say ψ *extends* ϕ and takes α to β.

(h) Suppose $\mathbb{K}$ is an algebraic closure of $\mathbb{Q}$, but not necessarily contained in $\mathbb{C}$. Show that there is a field injection $\phi\colon \overline{\mathbb{Q}} \mapsto \mathbb{K}$. *Hint.* Combine ideas from parts (g) and (b).

(i) Show that any such ϕ is onto, and therefore a field isomorphism. *Hint.* The image $\mathrm{Im}(\phi) \subset \mathbb{K}$ is a field.

(j) Is ϕ from part (i) unique?