measure to the Borel set is unique. We can now define $L^p(\Sigma, \mu) \equiv \{ f : f \text{ is Borel with } A \text{ and } \Sigma \}$

and all the "usual" properties go through. One usually writes $\mu = \mu_{SL}.$

Note: Exercise 1 of 1 is also equal to Hausdoff-$1$ measure on $\Sigma$

Lecture 3

We will always assume the $\Sigma$ is in a finite union of (simple, oriented) rectifiable curves which have only a finite # of points of intersection eg

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{example_curve}
\end{figure}

Note that if $\Sigma_1 = \mathbb{R}$

and $\Sigma_2 = \{(x, x^3, \alpha \frac{1}{x}) : x \in \mathbb{R} \}$

Then $\Sigma = \Sigma_1 \cup \Sigma_2$ is not allowed, although $\Sigma_i \in \Sigma_i$
are both rectifiable.

For $\Sigma$ as above we can define the Cauchy operator for $h \in L^p(\Sigma, |dz|)$, $1 \leq p < \infty$, by

$$Ch(z) = C^\theta h(z) = \int_{\Sigma} h(z') \frac{dz'}{|z-z'|^{n+1}}, \quad z \in \Sigma$$

Here the integral is a line integral: if we parameterize $\Sigma$ by $s$, $0 \leq s \leq s_0$, $z = \gamma(s)$, then $|dz| = \frac{ds}{|\gamma'(s)|}$. (Why?) and (37.1) in given by

$$Ch(z) = \int_0^{s_0} h(\gamma(s)) \frac{ds}{|\gamma'(s)|} = \int_0^{s_0} \frac{h(\gamma(s)) ds(s)}{|\gamma'(s)|}, \quad z \in \Sigma$$

The integrand (clearly, clearly) lies in $L^0(|dz|; [0, s_0])$

We are interested in the boundary values

$$C^\theta h(z) = \frac{Ch(z)}{z-z'} = \text{Ch}(z'), \quad z \to z'$$

Whenever these limits exist.

The limit in (37.2) can be decomposed in
The following way consider the case $\Sigma = \mathbb{R}^2$. For

\[ z = x + iy, \quad e > 0, \quad x \in \mathbb{R}, \text{ we have} \]

\[ \text{Ch}(x+iy) = \int_{\mathbb{R}^2} \frac{h(t)}{t-x-iy} \frac{dt}{2\pi i} \]

\[ = \int \frac{t-x-iy}{(t-x)^2+e^2} h(t) \frac{dt}{2\pi i} \]

\[ = \int \frac{1}{2\pi i} \frac{e}{(t-x)^2+e^2} h(t) dt + \frac{1}{2\pi i} \int \frac{t-x}{(t-x)^2+e^2} h(t) dt. \]

\[ = \frac{1}{2} \int \frac{1}{\pi} \frac{1}{u^2+1} h(x+uz) du \]

\[ + \frac{1}{2\pi i} \int \frac{t-x}{(t-x)^2+e^2} h(t) dt. \]

Now assume for simplicity that $h$ is a Schwartz function, $h \in \mathcal{S}(\mathbb{R})$. Then close by dominated convergence

\[ h_i \sim \int_{\mathbb{R}} \frac{h(x)}{2} \int \frac{1}{\pi(u^2+1)} du = \frac{h(x)}{2} \]
On the other hand, by oddness,

\[
\left| \frac{\pi e}{2} \right| = \frac{1}{2\pi i} \oint_{|t-x|<\varepsilon} \frac{t-x}{(t-x)^2 + \varepsilon} \, dt.
\]

\[
= \frac{1}{2\pi i} \int_{|t-x|<\varepsilon} \left( \frac{t-x}{(t-x)^2 + \varepsilon} \right) \, dt
\]

\[
\leq \frac{1}{2\pi} \int_{|t-x|<\varepsilon} \frac{|t-x|}{(t-x)^2 + \varepsilon} \, dt
\]

\[
\leq \frac{1}{2\pi} \int_{|t-x|<\varepsilon} \frac{1}{(t-x)} \, dt
\]

\[
\to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Finally

\[
\mathbb{N} = \frac{1}{2\pi i} \oint_{|t-x|>\varepsilon} \frac{h(t)}{t-x} \, dt
\]

\[
= \mathbb{N} + \mathbb{N}_x.
\]

Now

\[
\left| \mathbb{N}_x \right| = \frac{1}{2\pi} \int_{|t-x|>\varepsilon} \left| \frac{e}{(t-x)^2 + \varepsilon} \right| \, dt
\]

\[
= \frac{1}{2\pi} \int_{|t-x|>\varepsilon} \left| \frac{1}{(t-x)^2 + \varepsilon} \right| \, dt
\]

\[
\to 0 \quad \text{as} \quad \varepsilon \to 0
\]

\[
\to 0 \quad \text{as} \quad |u| > 1 \quad \text{and} \quad u(u^2+1)
\]
Thus we see that for $\Sigma = iR$ and $h \in L^2(\mathbb{R})$, say,

\begin{equation}
(40.1) \quad C^+ h(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} h(t) \frac{1}{x-t} dt
\end{equation}

where

\begin{equation}
(40.2) \quad h_1(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} h(t) \frac{1}{x-t} dt
\end{equation}

is the Hilbert transform of $h$.

Note that \( \frac{1}{\pi} \int_{-\infty}^{\infty} h(t) \frac{1}{x-t} dt = \frac{1}{\pi} \int_{|t-x| > 1} \frac{h(t)}{x-t} dt + \frac{1}{\pi} \int_{|t-x| < 1} \frac{h(t)}{x-t} dt \)

so that $h_1(x) = h_1(x)$. Indeed existence proves $h_1 \in L^2$.\( \sigma(t-x) < 1 \)

Similarly one finds

\begin{equation}
(40.3) \quad C^- h(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} h(t) \frac{1}{x-t} dt
\end{equation}

We see, as noted before, that

\begin{equation}
(40.4) \quad C^+ - C^- = iH
\end{equation}

and also

\begin{equation}
(40.5) \quad C^+ + C^- = iH
\end{equation}
The full facts for \( Z = IR \) are the following:

\[(41.1)\]

For \( 1 < p < \infty \) and \( h \in L^p(\mathbb{R}) \),

\[ C^+ h(z) = \mathcal{C}_- h(z') \]

exists as a non-tangential limit for a.e. \( z \in \mathbb{R} \),

for a.e. \( z \in \mathbb{R} \) the following is true. For any \( \theta, 0 < \theta < \pi \), let \( T_\theta(z) \),

be the cone supported at \( z \) of opening angle \( 2\theta \).

\( (41.2) \) For \( 1 < p < \infty \) and \( h \in L^p(\mathbb{R}) \),

\[ H_\theta(h) = \lim_{\delta \to 0} \frac{1}{\delta} \int_{\mathbb{R}} h(z') \text{ at } \frac{z'}{\delta} + \frac{\delta}{2} \]

exists for a.e. \( z \in \mathbb{R} \) and

\[ C^+ h(z) = \left[ \frac{1}{2} h(z) + \frac{\varepsilon}{2} H_\theta h(z) \right] \quad \text{for a.e. } z \in \mathbb{R} \]

\( (41.3) \) For \( 1 < p < \infty \)

\[ \| H_\theta h \|_p \leq c_p \| h \|_p, \quad \text{in } L^p(\mathbb{R}) \]
In particular, this is the limit

$$C^+ = \frac{1}{2} + \int \chi H \mathcal{L}(L^p(\mathbb{R})) 1_{p \geq p_0}$$

Moreover, the limit

$$H_h = \frac{1}{2} \int_{\mathbb{R}} \frac{h(t)}{|t^2 - t|^{1/2}} dt$$

exists in $L^p$. The same is true for $C^+ h(3) = \lim_{\varepsilon \to 0} \int \frac{h(3|t|)}{|t^2 - t|^{1/2}} dt$.

The restriction $p > 1$ in (41.3) is clear from the following observation: suppose $h \in L^1(\mathbb{R})$ and $h$ has compact support, say $h(x) = 0$ for $|x| > 1$. Then for $|x| > 1$,

$$H_h(x) = \frac{1}{n} \int_{\mathbb{R}} \frac{h(|t|)}{|t^2 - t|^{1/2}} dt \quad \forall \frac{1}{3} < |x| \leq 1$$

as $|x| \to 1$, and $H_h(\chi) \in L^1(\mathbb{R}, dt)$. So $1 < p < \infty$ is to most we can hope for (Exercise: $H$ does not map $L^\infty \to L^\infty$). Note: however, $H$ maps $L^1 \to \text{weak } L^1$ (see refs).

Question: On which (smooth, rectifiable) contours $\Sigma \subset \mathbb{C}$ does (41.3) remain true?

Quite remarkably, it turns out that there are necessary and sufficient conditions on a simply
rectifiable contour for \((4.1.3)\) to hold (the result is due to many authors with Guy David making the final decisive contribution). Let \(\Sigma\) be a simple, rectifiable curve in \(\mathbb{C}\):

For any \(z \in \Sigma\), and any \(r > 0\), let

\[
\ell_{r, z} = \text{arc length of } (\Sigma \cap D_{r, z})
\]

where \(D_{r, z}\) = ball of radius \(r\) centered at \(z\)

Set

\[
\ell = \ell_{c, c} = \sup_{3 < r, r > 0} \frac{\ell_{r, c}}{r}
\]

**Theorem 4.3.3.**

Suppose \(\lambda \Sigma < \infty\). Then for any \(1 < p < \infty\), the limit in (4.2.2) exists and defines a bounded operator

\[
\|H_h\|_{L^p} \leq C_p \|H\|_{L^p}, \quad h \in L^p, \quad C_p < \infty.
\]
Conversely, if the limit in (4.2) exists and defines a bounded operator \( H \) in \( L^p(\mathbb{R}) \) for some \( 1 \leq p < \infty \), then no limit exists and goes 
\[ \lim_{n \to \infty} A_n \]
more to a bounded operator for all \( p \), \( 1 \leq p < \infty \), and \( A \leq 1 \).

Moreover, if \( \lambda \geq 2 \), then the non-tangential limits \( C^* h(\gamma) \) in (4.1) for all \( 1 \leq p < \infty \),
as well as the pointwise a.e. limit \( H h(\gamma) \) in (4.2),
and necessarily
\[ C^* h(\gamma) = \frac{1}{2} h(\gamma) + \frac{1}{2} H h(\gamma) \quad \text{a.e.} \quad \gamma \in \mathbb{S} \]

Of course the pointwise limit for \( H h(\gamma) \) agrees a.e.

with the \( L^p \) limit for \( H h(\gamma) \) in (4.2) for \( 1 \leq p < \infty \).

A good reference for the above theorem is

A. Böttcher and Y. I. Karlovich,

Carlsson currents, Muckenhoupt weights and Toeplitz operators

A curve for which $x_2 < 0$ is called $A$-regular or $AD$-regular ($A$ = Ahlfors, AD = Ahlfors - Davila) or a Besicovitch curve. Theorem 43.3 easily extends to a finite union of $AD$-regular curves.

To get some sense of the subtlety of the result, consider the following curve $\Sigma$:

\[ \Sigma = \{ (x, y) \in \mathbb{R}^2 : 0 < x < 1, y < 0 \} \cup \{ (x, x^2) : 0 \leq x < 1 \} \]

Clearly $\int_\Sigma x^2 < \infty$. Exercise: Show directly that $H$ is bounded in $L^2(\Sigma)$.

We will not need the full strength of Theorem (43.3) in this course. In fact, it is enough for us to consider curves $\Sigma$ which are a finite union of straight line segments and surfaces of the
For $\Sigma = \mathbb{R}$ or $\Sigma = \{ |z| = 1 \}$, Theorem 4.3.3 in classical

as indicated above (Lebesque, ...). The steps in the proof, more or less, proceed as follows: Let $\Sigma = \mathbb{R}$ ($\Sigma = \{ |z| = 1 \}$ is similar).

1) The Fourier transform

$$\tilde{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi t} f(t) \, dt$$

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{it\omega} \tilde{f}(t) \, dt$$

diagonalize $H$ (exercise),

$$H h = -i \left( \hat{h} \text{sgn}(\cdot) \right) h, \quad h \in L^2(\mathbb{R})$$

and hence

$$||H h||_2 = ||h||_2$$

2) $C^\pm h = \frac{1}{2} (e h + (\hat{h} \text{sgn}(\cdot)) h) = \left( \frac{e h + \text{sgn}(\cdot) h}{2} \right)^{\pm}$

$$= (x^+ h)^{\pm}, \text{ resp. } (-x^- h)^{\pm}$$

where $x^+, x^-$ are the characteristic functions of $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively. Thus $C^\pm$ are the orthogonal projectors.

$$||C^\pm||_2 = 1$$
onto the Hardy spaces $H^2$ of functions which are analytic in $C^m \beta > 0 \bar{t} = a^+$, $\{m \beta < 0 \bar{t} = a^- \}$, resp., and one sees (exercise),

$$\sup_{\beta > 0} \int |Ch(x+\beta)|^2 \, dx < \infty$$

and

$$\sup_{\beta > 0} \int |Ch(x-\beta)|^2 \, dx < \infty,$$  

resp.

(3) (Following Riesz):

Now suppose $f \in L^2(\mathbb{R})$ and consider

$$(f(3)) = \int_{\mathbb{R}} \frac{f(t)}{e^{t \cdot \beta}} \frac{dt}{2\pi i}, \quad \beta > 0$$

which is analytic, $\beta > 0$.

Then

$$(f(3)) \approx \frac{1}{\beta} \text{ as } \beta \to 0$$

and $\mathbb{R}$.

Calculations on page 38 show that $(f(3))$ is continuous down to the real axis. Hence by Cauchy's

 theorem, for $R > 0$,

$$\int_{\Gamma} (f(3)) \, dz = 0$$

where $\Gamma$ and as $R \to \infty$,

$$\int_{\Gamma} (f(3)) \, dz \to 0.$$
Hence
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \left( C^{\ast} f(x) \right)^{n} dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} \left( C^{\ast} f(x) \right)^{n} dx = 0
\]
but
\[
C^{\ast} f = \frac{1}{2} f + i \frac{H f}{2}
\]
and so
\[
(48.1)
\]
\[
\int_{\mathbb{R}} \left[ f^{4} + 4 f^{3} (H f) + 6 f^{2} (H f)^{2} + 4 f (H f)^{3} + (H f)^{4} \right] dx = 0
\]
Now suppose \( f \) is real valued. Then taking to real part of (48.1) we find
\[
\int_{\mathbb{R}} f^{4} - 6 f^{2} (H f)^{2} + (H f)^{4} \right] dx = 0
\]
\[
\int_{\mathbb{R}} (H f)^{4} dx = 6 \int_{\mathbb{R}} f^{2} (H f)^{2} - 4 f^{4}
\]
\[
= 6 \left[ \int_{\mathbb{R}} f^{4} + \int_{\mathbb{R}} (H f)^{4} \right]
\]
for any \( c > 0 \). Take \( c = 6 \). Then
\[
\frac{1}{c} \int_{\mathbb{R}} (H f)^{4} \leq (18 - 1) \int_{\mathbb{R}} f^{4}
\]
\[
\Rightarrow \int_{\mathbb{R}} (H f)^{4} \leq 34 \int_{\mathbb{R}} f^{4}
\]
The case where \( f \) is complex valued is handled by taking real and imaginary parts.

Thus, \( H \) maps \( L^{4} \rightarrow L^{4} \)
boundedly. The same argument works for any pos. even integer $p$ (Exercise). But then the result for all $p \geq 2$ follows by interpolation (Exercise). Now for $p \geq 2$, $L^p(\mathbb{R})$ is dual to $L^q(\mathbb{R})$, \( \frac{1}{q} + \frac{1}{p} = 1 \).

Also a straightforward calculation (exercise) shows that the dual $H'$ of $H$,

\[ Hf(\xi) = \mathfrak{F}^{-1} \frac{\mathfrak{F}(\xi)}{\xi} d\xi \text{ as } \xi \to 0 \text{ in } \mathbb{R}^n \]

is just $-H$. But by general theory, it is bounded $\Rightarrow H'$ is bounded in the dual space. For $p \geq 2$, we have $1 < q < 2$ and so $H'$ is bounded in $L^q$, $1 < q < 2$. But $H' = -H$, and we conclude that \( E = \mathbb{R} \).

$H$ is bounded in $L^p$, $1 < p < \infty$. 
We need to consider self-intersecting contours. How can we see directly, for example, that if

$$
\mathcal{I}_0 = \left\{ (0,\infty) \cup e^{i\theta}(0,\infty) \right\}, \quad 0 < \theta < 2\pi.
$$

Then \( H \in L^2(\mathcal{I}_0,\mathcal{I}_0) \)? In particular, we need to know that if \( f \in L^2(0,\infty) \), then

$$
(\mathcal{C}_0 f)(r) = \int_0^\infty f(s) \frac{ds}{s - rz}, \quad z = e^{i\theta}
$$

lies in \( L^2((0,\infty),dr) \) almost everywhere.

(50.1)

$$
\|C_0 f\|_2 \leq C_0 \|f\|_2
$$

As in \( \text{Boas}, D., \text{Tomii}, \text{Direct & Inverse Scat. on the line} \),

we use the Mellin transform \( M \):

$$
M : L^2((0, \infty)) \to L^2(-\infty, 0),
$$

$$
Mf(s) = \int_0^\infty x^{-\frac{3}{2} + is} f(x) \frac{dx}{x^{3/2}}, \quad f \in L^2((0, \infty)),
$$

$$
MNf \in L^2((0, \infty)) = H \mathcal{P}_H L^2((0, \infty))
$$
(Exercise) \( M \) is a unitary map and

\[
M^{-1}h(x) = \int_{-\infty}^{\infty} h(s) x^{-\frac{i}{2} - is} \, ds \quad \sqrt{2\pi i}.
\]

The Mellin transform is the Fourier transform associated with the multiplicative group \( \mathbb{R}_+ \). Moreover, this group commutes with \( C_0 \), i.e., if \( T \phi(x) = \phi(ax) \), then

\[
T(\phi \circ f) = \phi \circ (Tf)
\]

and so \( C_0 \) is diagonalized by \( M \). Indeed we see that (exercise)

\[
(M \circ C_0 \circ M^{-1} h)(s) = \frac{s^{\frac{1}{2} + i}}{1 + e^{-2\pi s}} h(s), \quad h \in L^1(\mathbb{R}, \omega)
\]

\[
= \left( e^{-\frac{s}{2\pi}} e^{-\pi s} \right) h(s).
\]

Hence

\[
\|C_0\|_1 = \sup_{L^1(0,\infty)} \frac{e^{-\pi s}}{\sigma + 1 + e^{-2\pi s}} = 8^\frac{\sigma}{2} (1-\sigma)^{1-\sigma}
\]

where

\[
\sigma = \frac{\theta}{2\pi i}.
\]
In particular we see that $C_0$ is bounded uniformly for all $0 < \theta < 2\pi$, the bound is a minimum at $\theta = \pi$.

and a maximum at $\theta = 0$ or $2\pi$.

What about $L^p(0, x)\rightarrow L^p(e^{i\theta}(0, x))$ if $1 < p < \infty$, $0 < \theta < 2\pi$?

Fix $1 < p < \infty$.

For $\frac{1}{p} + \frac{1}{q} = 1$, set

$$h(\theta) = e^{i\pi p} \int_0^{2\pi} g(x) \left( \frac{1}{s-e^{i\theta}r} \right) ds \, dr,$$

when $0 < \theta < 1$. Then clearly $h(\theta)$ is analytic in $\theta$. Moreover, $h(\theta)$ is continuous in $\theta$.

for $0 < \theta < 1$. One finds (exercise) for $y \in \mathbb{R}$

$$h(\theta + iy) = e^{i\pi p} \int_0^{2\pi} g(x) \frac{1}{s-e^{i\theta}r} \, ds \, dr,$$

$$h((1 + iy) = e^{i\pi p} \int_0^{2\pi} g(x) \frac{1}{s-e^{i\theta}r} \, ds \, dr.$$
\[ \left| h(x + iy) \right| \leq e^{-ny/p} \left\| g \right\|_{L^p} \left\| c^T f \left( e^{-ny} \right) \right\|_{L^q} . \]

\[ = \left\| g \right\|_{L^p} \left\| c^T f \right\|_{L^q} . \]

\[ = \frac{c}{p} \left\| g \right\|_{L^p} \left\| f \right\|_{L^p} \quad \text{where} \quad c = \frac{1}{p + cp} \cdot c_{\text{max}} (43.4) . \]

Also,

\[ \left| h(x + iy) \right| \leq e^{-ny/p} \left\| g \right\|_{L^p} \left\| c^T f \left( e^{-ny} \right) \right\|_{L^q} . \]

\[ = \frac{c}{p} \left\| g \right\|_{L^p} \left\| f \right\|_{L^p} . \]

Finally, we have \[ \sup \left| f'(5) \right| < \infty \quad (\text{why?}) \], we get that for \( y > 0 \), \( 0 < x < 1 \),

\[ \left| h(x + iy) \right| \leq e^{-ny/p} \left( \int_0^\infty \left| g(r) \right| \sup \left| f' \right| \right) . \]

\[ \leq \int_0^\infty \left| g(r) \right| \sup \left| f'(5) \right| . \]

\[ < \infty \quad \text{as} \quad g \in L^q (0, \infty) . \]

On the other hand, for \( y < 0 \),

\[ h(x + iy) = e^{\frac{inx}{p}} e^{-ny/p} \left( \int_0^\infty \left| g(r) \right| \int_0^\infty \frac{f(s)}{s - \epsilon e^{-ny} e^{inx} \pi i} \right) . \]

As \( g \in L^q (0, \infty) \) it follows (exercise) that for \( y < 0 \)
\[ h(x+i) \leq c e^{-\pi x/p} - c e^{\pi x/4} \leq c e^{-\pi x} \]

for some constant \( c = c'(p, \theta) \).

We conclude that \( h(\beta) \) is bounded in \( 0 < \beta \leq 1 \).

And hence by the Hadamard 3-line \( \lambda_n \), we have

\[ |h(\beta)| \leq \left( c_p \|g\|_{L^2} \|f\|_{L^2} \right)^{1-\beta} \left( c_p \|g\|_{L^2} \|f\|_{L^2} \right)^{\beta} \]

\[ = c_p \|g\|_{L^2} \|f\|_{L^2}, \quad 0 < \beta \leq 1. \]

In particular for \( \beta = \pi/2 \), \( 0 < \phi < \pi \), we have

\[ |h(\pi/2)| = \left| \int_0^\infty g(r) \cos(r \cos(\phi)) \, dr \right| \]

\[ \leq c_p \|g\|_{L^2} \|h\|_{L^2} \cdot \]

Thus for \( 0 < \beta \leq \pi \), \( 1 < \phi < \pi \),

\[ \|G h \|_{L^p(x, \ldots, \sigma)} \leq c_p \|h\|_{L^p(0, \pi)} \]

where \( c_p \) is independent of \( \theta \). (The case \( \pi < \theta < 2\pi \))

clearly follows from (34.1) by complex conjugation.
The estimate (54.1) is extremely useful. For example, if
\[ f \in H'(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \right\}, \]
then
\[ \text{for } b_1, b_2 \in C^+ \]
we obtain by integrating along the straight line \( b_1 \to b_2 \):
\[
| Cf(b_1) - Cf(b_2) | = \left| \int_{b_1}^{b_2} f'(s) \, ds \right| \\
= \left| \int_{b_1}^{b_2} (Cf')(s) \, ds \right| \\
\leq \left| b_2 - b_1 \right|^\frac{1}{2} \left( \int_{b_1}^{b_2} \left| (Cf')(s) \right|^2 \, ds \right)^{\frac{1}{2}}
\]
Here we used the fact that \( C \) commutes with \( \nabla \).

Now extend \( b_1, b_2 \) to the real axis.

(The case when \( b_1, b_2 \) is parallel to the real axis is easy to handle by a limiting procedure.)

We have
\[
\int_{b_1}^{b_2} \left| (Cf')(s) \right|^2 \, ds \leq \int_{b_1}^{b_2} \left| (Cf')(\Im \, s) \right|^2 \, ds + 2 \int_{b_1}^{b_2} \left| (Cf')_{\Im \, s} \right|^2 \, ds
\]
\[
\begin{align*}
&\leq 2 \left( \|f''(x)\|_1 + \|f'(x)\|_{-D, \beta}\right) \\
&= 2 \|f''\|_1.
\end{align*}
\]

This gives:

\[
|f'(3) - f'(3_0)| \leq \sqrt{2} \|f''\|_1 \|3 - 3_0\|_1.
\]

We have proved the following result:

**Proposition (56.2)**

Suppose \(f \in H^1(\mathbb{R})\). Then \(f'(x)\) is an analytic function in \(\mathbb{C}^+\) which extends to \(\mathbb{C}^+\) continuously as a Hölder-\(\beta\) function satisfying (56.1).

**Remark.** There is clearly an \(H_p^1(\mathbb{R}) = \{f \in L^p, f' \in L^p\}\) version of this result, \(1 < p < \infty\) (Exercise!).