In our discussion of the Wiener-Hopf technique (lectures 6-11; see also Remark on 10.1-10.2) the following issue arose: if we know that the normalized RHT $\gamma$ has a solution $u(t) \in \mathcal{C}(\mathbb{R})$, how much do we know about $(1-\gamma)u$? If we wrote $u(t) = f + h(t), h(t) \in \mathcal{C}(\mathbb{R})$ then

$$h(t) = u(t) - f$$

where $f = u(t) \in L^p(\mathbb{R})$. So if we know that the normalized RHT $\gamma$ has a solution, we know that the RHT $\mathcal{R}_2$ has a solution (only) for the special RHS $F = u(t)$. How much more do we need to know about $u(t)$ to conclude that

$$1-\gamma$$

is a bijection? The following result addresses this question.
Theorem 157.1
Let \( u, \Sigma \) be as above, with \( u - i \in L^p(\Sigma) \).

Let \( C_0 = C^- (\Sigma, u) \) be the operator corresponding to the (trivial) pointwise factor such that
\[
L = (1 - C_0)^{-1} u_v \quad \text{and} \quad u_v = u, \quad u - i = +.
\]

Then

1. \( C_0 \) is a bijection.

The normalized RHP \((\Sigma, v)\) has a solution \( m_+ \in \Sigma + \mathcal{O}(C(\Sigma)) \) such that \( m_+ \in \Sigma + \mathcal{O}(C(\Sigma)) \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( 1 \leq p, q < \infty \).

The map
\[
T_h = (C^+ h(m_+)^{-1}) m_+
\]

is bounded in \( L^p(\Sigma) \).

Remark 157.2: Suppose \( \Sigma \) is the classical sense, and \( \det m_+ \neq 0 \), then \( 1 - C_0 \) is invertible in \( L^p(\Sigma) \) for all \( 1 < p < \infty \).

Proof: Exercise.

Proof of Theorem 157.1
Suppose \( 1 - C_0 \) is a bijection. Then

the equation \( (1 - C_0) \mu = I \) has a unique
solution $\mu = I + L^p(\Sigma)$. More precisely, if $\mu = I + h$, then $(1 - (\varphi)^\perp)h = F$, $F = C_{\varphi}I$, which implies $L^p(\Sigma)$ has a unique solution with $h \in L^p(\Sigma)$. Standard computations show that

$$m^* = I + C^*(\mu(I - I))$$

solves the normalized MTP $(\Sigma, \nu)$ in $L^p$.

Now for row vectors $h \in L^p$, its dual space consists of row vectors $g \in L^q$, such that $\frac{1}{p} + \frac{1}{q} = 1$, and its pairing of $L^p$ and $L^q = (L^p)^*$ may be realized by

$$\langle h, g \rangle = \int h(\xi)g(\xi) + \delta_{\xi}$$

as

$$C^* = \frac{1}{2}H + \frac{1}{2}H$$
where \( H \) is the Hilbert transform

\[
H h(s) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \frac{h(s)}{s-s'} ds', \quad s \neq 0
\]

we see that in the pairing (158.1)

\[
(C^+) = \frac{1}{2} \text{Re} + \frac{i}{2} \text{Im} H
\]

\[
= \frac{1}{2} - \frac{i}{2} \text{H}
\]

\[
= - (-\frac{1}{2} + \frac{i}{2} \text{H})
\]

\[
= - C^-
\]

and similarly

\[
(C^-)' = - C^+
\]

Of course \( C^+ \) is \( L^1 \).

Now by general theory

\[
1 - C = \text{a bijective in } L^p
\]

\[
(1 - C)' = \text{a bijective in } L^q.
\]

We have \( C v = C^+ R_{v-I} \quad \text{when } R_{v-I} \)

denotes right multiplication by \( v-I \), i.e.

\[
R_{v-I} h = h(v-I)
\]
Also
\[
\langle R_{\nu^{-1}} h, g \rangle = h(\nu^{-1} g^T)
\]
\[
= h \left( R_{\nu^{-1}} g \right)^T.
\]
\[
= \langle h, R_{\nu^{-1}} g \rangle.
\]

so \( R_{\nu^{-1}} = R_{\nu^{-1}} \). Hence
\[
(1 - C) = (1 - C R_{\nu^{-1}}) = 1 + R_{\nu^{-1}} C^+ \]
\[
= 1 + R_{\nu^{-1}} C^+.
\]

But as noted before, by general theory

\( 1 + R_{\nu^{-1}} C^+ \) is a bijection.

\[
\Rightarrow
\]

\( 1 + C^+ R_{\nu^{-1}} \) is a bijection.

We have for \( g \in L^1(\mathbb{S}) \).
\[
(1 + C^+ R_{\nu^{-1}}) g = g + C^+ g(\nu^{-1} - I)
\]
\[
= g + C^{-} g(\nu^{-1} - I) + g(\nu^{-1} - I)
\]
\[
= (g \nu^T) - C^{-} g \nu^T (\nu^{-1} - I)
\]
\[ = (1 - C_{v^{-T}}) \circ R_{v^{I}} \circ \sigma \]

The above calculations imply that

\[ 1 - C_{v^{-T}} \text{ is a bijection in } L^p(\Sigma) \]

\[ 1 - C_{v^{-T}} \text{ is a bijection in } L^q(\Sigma) \]

In particular, we can conclude as above that a normalised \( \mathcal{N^{+I}}(\Sigma, v^{-T}) \) in \( L^q \)

has a unique solution \( \tilde{m}_T \in I + \text{DC}(L^q) \)

\[ \tilde{m}_T = \tilde{m} - v^{-T} \]

But note that

\[ m_T \tilde{m}_T^T = m_T - v^{I} \cdot \tilde{m}_T = m_T \tilde{m}_T \]

and by familiar arguments we see that \( \tilde{m}_T = m_T^{-1} \)

on \( \Sigma \) and no

\[ m_T^{-1} \in I + \text{DC}(L^q) \]

Finally, for any \( F \in L^p(\Sigma) \) we have \( \tilde{m}_T \in I + \text{DC}(L^p) \)

\[ \tilde{m}_T = m_T - v^{-T} + F \]
has a unique solution \( m_\pm \in \mathcal{O}(L^p) \) and

\[
\| m_\pm \|_{L^p} \leq c \| F \|.
\]

(162.2)

(See p.157.) Setting \( v = m^{-1}_+ \) in (161.1), we obtain

\[
K_+ m_+^{-1} = K_- m_-^{-1} + F m_+^{-1}.
\]

Hence

\[
K_+ = (C^+ F m_+^{-1}) m_+.
\]

Note: \( K_- = C (F v^{-1} m_-) m_- \) does not exist \( v, v^{-1} \in L^p \)

It follows then by (162.2) that \( T \), in particular, is bounded in \( L^p(\Sigma) \), as desired.

Conversely, suppose that \( m_\pm \in \mathcal{I} + \mathcal{O}(L^p) \) solves the normalized RWP \((\Sigma, \delta^-)\) with \( m_\pm^{-1} \in \mathcal{I} + \mathcal{O}(L^p)\) and

\[
T h = (C^+ h(m_+)^{-1}) m_+ \text{ bounded in } L^p.
\]

Consider the solution

\[
(162.3) \quad \begin{cases}
K_+_+ = K_- v^- + F, & F \in L^p \\
m_\pm \in \mathcal{O}(L^p)
\end{cases}
\]
Assume first that \( F \in L^1 \cap L^\infty \subset C L^p \) and set

\[ M_\pm = (C^\pm (F m_\pm')) m_\pm \]

Clearly

\[ M_\pm = TF \]

and

\[ m_\pm' = (C^\pm (F m_\pm')) m_\pm \]

\[ = (C^+ (F m_\pm') - F m_\pm') m_\pm \]

\[ = TF - F \]

It follows that

\[ \|M_\pm\|_{L^p} \leq C \|F\|_{L^p} \]

Let \( h = F m_\pm' \). As \( F \in L^1 \cap L^\infty \) and \( m_\pm' \in I+OC(L^q) \), it follows that \( h \in L^1 \cap L^\infty + L^q \subset C L^q \). But

Then as \( m_\pm \in I+OC(L^p) \), it follows that

\[ (C^\pm (F m_\pm')) m_\pm = (C^\pm h) m_\pm = C^\pm k \]

where \( h \in L^q + L^1 \). But \( k = C^+ h - C^- k \)

\[ = M_+ - M_- \] so that in fact \( k \in L^p \). Hence

\[ \|m_\pm\|_{C^1} \leq OC(L^p) \]
\[ M_+^{-1} = C^+ (F M_+^{-1}) = C^- (F M_-^{-1}) + E M_+^{-1} = H_+ M_+^{-1} + E M_-^{-1} \]

and so

\[ M_+ = E M_+^{-1} M_+ + F = E M_+ - F. \]

Thus for \( E \in L^\infty \cap C \) we have a unique (by 164.3)

solution of the \( IR^{+2} \)

\[
(164.1) \quad \left\{ \begin{array}{l}
E M_+ = M_+ - F \\
M_+ \in OC(L^p) \quad \| M_+ \|_{L^p} \leq C \| F \|_{L^p}.
\end{array} \right.
\]

In the proof we showed

\[ M_+ = C^{\pm} h = C^{\pm} (M_+ - M_-) \]

\[ = C^{\pm} (T F - (TF - F) \sigma^n) \]

Thus given \( F \in L^p \), where \( E_0 \in L^\infty \cap C \), \( F_0 - F \in L^p \),

we conclude that (164.1) can be solved for any \( F \in L^p \). In particular, \( 1 - \sigma^n \) is a bijection in

\[ L^p. \]
Now what is the general relationship between

\[ 1 - C_n \] and the normalized RHP \( \Sigma, \nu \) in

the case that \( 1 - C_n \) is not a bijection?

Consider the following simple example. Let

\[ \Sigma = \{ \{1, 1\} \}, \text{ and let } \nu = \delta^n \text{ on } \Sigma, \]

\[ n \in \mathbb{Z}. \]

Does a solution of the normalized RHP \( F \),

\[ m_+ = m_- \nu, \]

\[ m_+ \in \mathcal{I} + \mathcal{OC}(L^2), \]

Suppose \( n > 0 \):

\[ m_+ = m_- \delta^n. \]

Then we see that if \( m_{(3)} \) is the extension of

\[ m_+ \text{ off } \Sigma, \]

then

\[ \Sigma(1) = m_{(3)} \]

\[ B(1) \]

\[ = m_{(3)} \delta^n, \]

\[ \{1\} > 1. \]

is entire. But as \( m_{(3)} = 1 + \sum_{\delta^k \delta^{(1)}} \), \( h \in L^2 \),
we see that \( E(3) = m(3) s^n = s^n + O\left( \frac{1}{s^{n-1}} \right) \)

as \( s \to \infty \), and hence \( E(3) \) is a monic \( n \)-monic polynomial

of degree \( n \), \( E(3) = s^n + \ldots = p(3) \)

Hence

\[
m(3) = \frac{p(3)}{s^n}, \quad |z| > 1.
\]

Now if we require further that \( m^{-1} \in I + O(1) \),

then \( m(3) \) cannot have any zeros in \( \{ |z| \leq 1 \} \)

or in \( \{ |z| \geq 1 \} \). But \( p(3) \) has \( n \) zeros. This

is a contradiction. Hence \((165.1)\) cannot have a

solution with \( m \) and \( m^{-1} \in I + O(1) \), if \( n > 0 \).

Suppose \( n = -\tilde{n} < 0 \). Then

\[
E(3) = m(3) \quad (3) \leq 1
\]

\[
= m(3) s^n = m(3) s^n, \quad |z| > 1.
\]

But then \( E(3) \) is entire and as \( n \to -\infty \), \( E(3) = O\left( \frac{1}{s^n} \right) \).

Thus \( E(3) = 0 \) and so \( m(3) = 0 \), in particular \( m(3) \) is \( I \) as \( s \to -\infty \).
Thus in both cases a solution \( m_n \) of the normalized 
RHP \((\mathcal{E}, \mathcal{V})\) does not exist with \( m_n, m_n' \in 1 + \mathcal{O}(\mathcal{E}) \).

Notice however that a solution of the following

problem \( \mathcal{F} \):

\[(6.7.1)\]

\[2^k m_k = m_{-k} \]

when \( k \in \mathbb{Z} \), and \( m_k, m_k' \in 1 + \mathcal{O}(\mathcal{E}) \).

Indeed take \( k = n \) ad \( m_n = 1 \).

Now consider the RHP \( \mathcal{P}_{1,n} \) corresponding to \( \nu = s^n \) on

\[ \Sigma = \{ \mid \nu \mid = 1 \} \cdots \]

\[(6.7.2)\]

\[m_+ = \gamma_n s^n + F, \quad F \in L^2(\mathbb{Z}) \]

\[\kappa_+ \in \mathcal{O}(\mathcal{E}) \].

Suppose \( n \geq 0 \). In this case, the solution of \((6.7.2)\)

cannot be unique. Indeed if

\[\hat{\gamma}_n(1) = 1, \mid \hat{\gamma}_n(1) \mid = s^{-n}, \mid \hat{\gamma}_n(-1) \mid = s^n \]

Then if \( k \) solves \((6.7.2)\), then so does \( k + m_k \).
How non-unique is $M_\pm$? This is clearly the

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Now if

$$M_+ = m_{-3}^n, \quad M_- = c^2 h, \quad h \in L^2(E)$$

then

$$E(z) = M(z), \quad z \in (\xi^*)_{n-1}$$

is again a polynomial $p_{\xi}$ by now of degree $n-1$.

Here

$$M(\xi) = p_{\xi}, \quad |\xi| < 1$$
$$M(\xi) = \frac{p_{\xi}}{\xi^n}, \quad |\xi| > 1$$

for any $p_{\xi}$ of degree $\leq n-1$, gives a element of $N$.

Hence

$$\dim N = n$$

On the other hand given any $F \in L^2$, write

$$F = C^+ F - C^- F = F_+ - F_-$$

We seek $M_+ = M_{-3}^n + F_+ - F_-$
Thus

\[ M_{\pm} \cdot F = M_2 \cdot 3^n - F \]

It follows then as before that

\[ E(3) = M(3) - CF(3), \quad |\beta| < 1 \]

\[ = M_2 \cdot 3^n - CF(3), \quad |\beta| > 1 \]

\[ E(3) \text{ is entire \ and is of order } O(3^{-1}) \quad \text{as } n \to \infty \]

Thus \[ E(3) = \beta(3) \quad \text{for some poly. } p(3) \text{ of degree } \leq n - 1 \]

\[ M(3) = CF(3) + M(3), \quad |\beta| < 1 \]

\[ = \Theta(3) + \beta(3) \quad \text{for any poly. } p(3) \text{ of } \deg \leq n - 1 \]

\[ M_{\pm} = M_2 - v + F, \quad M_2 \in OC(3) \]

In terms of the relations of the operator \( 1 - \nu \) on \( L^2(\mathbb{R}^d) \), the above calculations show (Exercise)
\[
\dim \ker (1 - (1 - \nu)) = n
\]
\[
\dim \text{coker} (1 - (1 - \nu)) = 0
\]
In particular, \((1 - \nu)\) is Fredholm and
\[
\text{index} (1 - (1 - \nu)) = n - 0 = n
\]
Now consider (167.2) with \(n < 0\); \(\tilde{\nu} = -n > 0\).

Suppose
\[
\hat{\nu}_1 = \hat{\nu}_2 = \tilde{\nu}^n, \quad \tilde{\nu} = \mathcal{O}(L^2),
\]

Then as before
\[
E(3) = \hat{\nu}_1 (3), \quad \mathcal{O}(1)
\]
\[
= \hat{\nu}_1 (2), \quad \mathcal{O}(1)
\]

is entire and of order \(O(3^{-\tilde{\nu}+1})\) as \(\nu \to \infty\). Hence
\[
\hat{\nu} = 0 \quad \text{Thus}
\]
(170.1): \( \dim N = 0 \)

Now consider
\[
M_+ = M_- \beta^{-\tilde{\nu}} + F, \quad F \in \mathcal{L}^2, \quad M_+ \in \mathcal{O}(L^2)
\]
Again set \( F_{\pm} = C^{\pm}F \), \( F_{+} - F_{-} = F \).

Thus we must consider

\[
M_{\pm} - E_{\pm} = M_{\mp}^{2^{\pm}} - C^{-}F
\]

from which we see that

\[
E_{\pm} = M_{\pm}^{2^{\pm}} - C^{-}F, \quad |B| > 1
\]

is entire. As \( z \to \infty \), \( E_{\pm} \to 0 \). Hence \( E_{\pm} \equiv 0 \).

In particular

\[
M_{\pm}^{2^{\pm}} = C^{-}F, \quad \forall |B| > 1.
\]

\[
M_{\pm}(z) = 3^{2^{\pm}} C^{-}F(z) = 3^{2^{\pm}} \int_{\Sigma} \frac{F(s)}{s-z} \, ds
\]

\[
= -3^{2^{\pm}-1} \int_{\Sigma} F(s) \frac{1}{s-z} \, ds
\]

\[
= -3^{2^{\pm}-1} \int_{\Sigma} F(s) \left( 1 + \frac{s}{2} + \frac{s^2}{3} + \ldots + \frac{s^{2^{\pm}-2}}{2^{\pm-2}} \right) \, ds
\]

\[
+ O\left( \frac{1}{s} \right), \quad \text{as} \ z \to \infty
\]

but \( M_{\pm}(z) = O\left( \frac{1}{s} \right) \), so we must have

\[
(175.1) \quad \int_{\Sigma} F(s) \, ds = 0, \quad \text{as} \ |z| \to \infty, \quad |z|^2 = n-1, = |z|-1
\]
In terms of the relation of the operator \( I - \psi \) to the \( \mathfrak{H} \), the above calculations now show (exercise)

\[
\dim \ker (I - \psi) = 0
\]

\[
\dim \text{coker} (I - \psi) = \ln 1 = -n.
\]

In particular, \( I - \psi \) is again Fredholm, as

\[
\text{index } (I - \psi) = 0 - \ln 1 = n
\]

Of course if \( \psi \) is constant \( n = 0 \), we have again

\[
\dim \ker (I - \psi) = 0
\]

\[
\text{dim coker } (I - \psi) = 0
\]

\[
\text{and } \ln (I - \psi) = 0 = n.
\]

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**Exercise**

Consider the following matrix example on \( \Sigma = \mathfrak{H}(b_1=1) \): 

\[
\begin{pmatrix}
3 & 1 \\
0 & 3
\end{pmatrix}
\]

(i) Verify that \( \psi(b) \) has a factorization of the form