Lecture 9

Proof of Th^m 103.2

Let A_1 be a closed sym. exten. of A. From Lemma 103.1 we know $D(A_1) = D(A) \oplus_A S_1$ where S_1 is an A-closed, A-symmetric subspace of $K_+ \oplus_A K_-$. If $\varphi \in S_1$, it can be written uniquely as $\varphi = \varphi_+ + \varphi_-$. Since S_1 is A-sym,

$$
\begin{aligned}
0 &= (A^*\varphi, \varphi) - (\varphi, A^*\varphi) \\
&= (A^*(\varphi_+ + \varphi_-), \varphi_+ + \varphi_-) - (\varphi_+ + \varphi_-, A^*(\varphi_+ + \varphi_-)) \\
&= (i\varphi_+ - i\varphi_-, \varphi_+ + \varphi_-) - (\varphi_+ + \varphi_-, i\varphi_+ - i\varphi_-) \\
&= i\Big[-(\varphi_+, \varphi_+) + (\varphi_-, \varphi_-) + \cancel{(\varphi_-, \varphi_+)} - \cancel{(\varphi_+, \varphi_-)} \\
&\qquad - (\varphi_+, \varphi_+) + (\varphi_-, \varphi_-) + \cancel{(\varphi_+, \varphi_-)} - \cancel{(\varphi_-, \varphi_+)}\Big] \\
&= 2i(\varphi_-, \varphi_-) - 2i(\varphi_+, \varphi_+)
\end{aligned}
$$

which implies

(105.1) $$\|\varphi_-\| = \|\varphi_+\|$$

Let $B_1 \subset K_+$ denote the range of the projection of S_1 into K_+,

$$S_1 \ni \varphi = \varphi_+ \oplus \varphi_- \to \varphi_+$$

Clearly B_1 is a linear subspace of K_+. Now φ_- is uniquely determined by φ_+. For if $\varphi_+ \oplus \varphi_-$ and $\varphi_+ \oplus \tilde{\varphi}_-$ both belong to S_1, then

$$\varphi_- - \tilde{\varphi}_- = (\varphi_+ \oplus \varphi_-) - (\varphi_+ \oplus \tilde{\varphi}_-) \in S_1 \quad \text{i.e.}$$

$0 + (\varphi_- - \tilde{\varphi}_-) \in S_1$ & hence by (105.1)

$$\|\varphi_- - \tilde{\varphi}_-\| = \|0\| = 0 \qquad \text{so} \qquad \varphi_- = \tilde{\varphi}_-$$

Set
$$U\varphi_+ = \varphi_- \quad , \quad \varphi_+ \in B_1$$
$$U\varphi_+ = 0 \qquad \text{if } \varphi_+ \perp B_1 \text{ in } K_+ .$$

$\boxed{\text{Insert } 106.1}$

Then by (105.1), U is a partial isometry from K_+ to K_- with initial space $I(U) = B_1$. We have

$$D(A_1) = \{\varphi + \varphi_+ + U\varphi_+ : \varphi \in D(A), \varphi_+ \in I(U)\}$$

and as $A_1 \subset A^*$

$$A_1(\varphi + \varphi_+ + U\varphi_+) = A^*(\varphi + \varphi_+ + U\varphi_+)$$
$$= A\varphi + i\varphi_+ - iU\varphi_+$$

Conversely, let U be an isometry from a subspace

Insert on p 106

Note that B_1 is a closed subspace of K_+.

Indeed suppose that $f_n \in B_1$ and $f_n \to f \in K_+$. We must show $f \in B_1$ i.e. $f = \varphi_+$ for some $\varphi \in S_1$. Now

$f_n = (\varphi_n)_+$ for some $\varphi_n = (\varphi_n)_+ + (\varphi_n)_- \in S_1$

As $\{(\varphi_n)_+ = f_n\}$ is Cauchy, $\{(\varphi_n)_-\}$ must also be Cauchy by (105.1); we have $(\varphi_n)_- \to g \in K_-$. Hence

$\varphi_n = (\varphi_n)_+ + (\varphi_n)_- \to f + g$. Also $\varphi_n \in K_+ \oplus_A K_-$

$\subset D(A^*)$ & $A^*\varphi_n = A^*(\varphi_n)_+ + A^*(\varphi_n)_-$

$$= i((\varphi_n)_+ - (\varphi_n)_-)$$

$$\to i(f - g)$$

Hence as A^* is closed, $\varphi = f + g \in D(A^*)$ & $A^*\varphi = A^*(f+g)$

$= i(f-g)$. But $(\varphi - \varphi_n, \varphi - \varphi_n)_A = \|\varphi - \varphi_n\|^2$

$+ \|A^*\varphi - A^*\varphi_n\|^2 \to 0$ and so, as S_1 is A-closed,

it follows that $\varphi \in S_1$. But then $f = \varphi_+$, as desired.

of K_+ into K_- and define $D(A_1)$ and A_1 by (104.1) and (104.2) resp. Then it is easy to check (exercise) that $D(A_1)$ is an A-closed, A symmetric subspace of $D(A^*)$, and so by Lemma 103.1, A_1 is a closed symm. extension of A.

For $u \in D(A_1)$, $u = \varphi + \varphi_+ + U\varphi_+$

(107.0) $$(i - A_1)u = (i-A_1)\varphi + (i-A_1)(\varphi_+ + U\varphi_+)$$

$$= (i-A)\varphi + 2i\,U\varphi_+.$$

Now recall that

(107.1) $$\ker(i + A^*) = \operatorname{Ran}(i-A)^\perp.$$

$\cancel{\boxed{A_s \quad A^* \subset A^+}}$ We have $(i+A^+)U\varphi_+ = (i-i)U\varphi_+ = 0$ so $U\varphi_+ \in \ker(i+A^+)$ and so in particular, $U\varphi_+ \perp (i-A)\varphi$

Thus

(107.2) $$\operatorname{Ran}(i-A_1) = \operatorname{Ran}(i-A) \oplus \operatorname{Ran} U)$$

and so if $\dim I(U) = \dim \operatorname{Ran} U < \infty$, we conclude from (107.1) that $\operatorname{ran}(i-A)^{\perp} = (\operatorname{ran}(i-A_1))^{\perp} \oplus (\operatorname{ran} U)$ ..

$$n_-(A_1) = n_-(A) - \dim I(U)$$

Similarly $n_+(A_1) = n_+(A) - \dim I(U)$.

This proves the Theorem. $\square$

The following result is immediate.

Corollary (108.1)

Let A be a closed, symmetric operator with deficiency indices n_+ and n_-. Then,

(a) A is s. adj $\iff n_+ = n_- = 0$

(b) A has s. adj. extensions if and only if $n_+ = n_- \geq 0$. There is a 1-1 correspondence between s. adj. extensions of A and unitary maps from K_+ onto K_-

(c) If either $n_+ = 0 \neq n_-$ or $n_- = 0$ and $n_+ \neq 0$ then A has no non-trivial symmetric extensions (such operators are called maximal symmetric).

Proof : (a) follows from Th^m 55.1

(b) If A_U is a s.adj. extension of A, then from (107.2)

$$\mathrm{ran}\,(i - A_U) = \mathrm{ran}(i-A) \oplus \mathrm{ran}\, U$$

But as A_U is s.adj, $\mathrm{ran}\,(i - A_U) = \mathcal{H}$ and so

$\mathrm{ran}(i-A) \oplus \mathrm{ran}\, U = \mathcal{H}$. In other words

(109.1) $$\mathrm{ran}\, U = \mathrm{ran}(i-A)^{\perp} = \ker(i + A^*)$$

Similarly we find

$$\mathcal{H} = \mathrm{ran}\,(i + A_U) = \mathrm{ran}(i+A) \oplus I(U)$$

and so

$$I(U) = \mathrm{ran}(i+A)^{\perp} = \ker(i - A^*)$$

As $\dim I(U) = \dim \mathrm{ran}\, U$ we see that

(109.2) $$n_+ = \dim \ker(i - A^*) = \dim \ker(i + A^*) = n_-$$

Thus $n_+ = n_-$, whether finite or not. Conversely, suppose that $n_+ = n_-$, finite or not. Let U be a unitary map from K_+ onto K_-, which is possible as $n_+ = n_-$ and let A_U be the associated symm. extension of A. Then from (107.2), $\mathrm{ran}(i - A_U) = \mathrm{ran}(i-A) \oplus \mathrm{ran}\, U = \mathrm{ran}(i-A) \oplus K_-$ $= \mathrm{ran}(i-A) \oplus \mathrm{ran}(i-A)^{\perp} = \mathcal{H}$. Similarly $\mathrm{ran}(i + A_U) = \mathcal{H}$. Hence A_U is s.adj. This proves (b).

$\boxed{\cancel{K_+ \text{ onto } K_-}}$

(c) Clearly if $n_+ = 0 \neq n_-$ or $n_+ \neq 0 = n_-$ there are no (non-trivial) partial isometries U. D.

We now illustrate Th^m 103.2 with the example

$$D(T) = \{ f = L^2(0,1) : f \in AC,\ f' \in L^2,\ f(0) = f(1) = 0 \}$$

$$Tf = if' \qquad , \ f \in D(T)$$

We showed earlier that T is a closed, sym operator and all its self-adj. extensions S_α have the form

$$D(S_\alpha) = \{ f \in L^2(0,1) : f \in AC,\ f' \in L^2,\ f(1) = e^{i\alpha} f(0) \}$$

$$S_\alpha f = if', \qquad f \in D(S_\alpha)$$

where $\alpha \in \mathbb{R}$.

We also showed

$$D(T^*) = \{ f \in L^2 : f \text{ AC}, f' \in L^2 \}$$

$$T^* f = i f', \quad f \in D(T^*).$$

Now $f_+ \in K_+ \iff (i - T^*) f_+ = 0$

$\iff f_+ - f_+' = 0 \implies f_+ = c e^x \quad \therefore K_+ = \{ c e^x : c \in \mathbb{C} \}$

Similarly $f_- \in K_- \implies f_- = c e^{-x} \quad \therefore K_- = \{ c e^{-x} : c \in \mathbb{C} \}$

Thus $n_+ = n_- = 1$ and T has s. adj extensions.

Let $\varphi_+ = \dfrac{\sqrt{2}}{\sqrt{e^2 - 1}} e^x$ and $\varphi_- = \dfrac{\sqrt{2}\, e}{\sqrt{e^2 - 1}} e^{-x}$

Note that $\int_0^1 |\varphi_+|^2 dx = 2 \int_0^1 \dfrac{e^{2x}}{e^2 - 1} dx = 1$

Similarly $\int_0^1 |\varphi_-|^2 dx = 1$, so $\varphi_\pm$ are normalized vectors in $K_\pm$ resp. Then the only isometries of K_+ onto K_- are maps of the form $\varphi_+ \to \gamma \varphi_-$ where $|\gamma| = 1$. Thus the s. adj. extension A_γ assoc.

with γ has

$$D(A_\gamma) = \{\varphi + \beta\varphi_+ + \beta\gamma\varphi_- : \varphi \in D(T), \beta \in \mathbb{C}\}$$

$$A_\gamma(\varphi + \beta\varphi_+ + \beta\gamma\varphi_-) = i\varphi' + i\beta\varphi_+ - i\beta\gamma\varphi_-$$

$$= i\varphi' + \beta i\varphi_+' + \beta\gamma i\varphi_-'$$

$$= i\frac{d}{dx}(\varphi + \beta\varphi_+ + \beta\gamma\varphi_-)$$

i.e.

(112.1) $$A_\gamma = i\frac{d}{dx} \quad \text{on} \quad D(A_\gamma).$$

Finally note that if $\psi \in D(A_\gamma)$, then

$$\psi(0) = \beta\frac{\sqrt{2}}{\sqrt{e^2-1}} + \beta\gamma\frac{\sqrt{2}e}{\sqrt{e^2-1}} = \frac{(1+\gamma e)\sqrt{2}\beta}{e^2-1}$$

and

$$\psi(1) = \frac{\sqrt{2}\beta(e+\gamma)}{\sqrt{e^2-1}}$$

so that

$$\psi(1) = \frac{e+\gamma}{1+\gamma e}\psi(0) = \alpha\psi(0)$$

where $$|\alpha| = \left|\frac{e+\gamma}{1+\gamma e}\right| = \left|\frac{e+\gamma}{\gamma^{-1}+e}\right| = \left|\frac{e+\gamma}{\overline{e+\gamma}}\right| = 1$$

Conversely, if $\psi(1) = \alpha\psi(0)$, then $\psi = \varphi + \beta\varphi_+$ (exercise) $+ \gamma\beta\varphi_-$ for some β where $\gamma = \alpha - e/1-\alpha e$. Then $A_\gamma = S_\alpha$.

Thus $\alpha \mapsto \gamma = \frac{\alpha - e}{1 - \alpha e}$ maps the "physical" circle to v. Neumann's abstract circle.

We note the following important result.

Definition

An antilinear map $C : \mathcal{H} \to \mathcal{H}$,

$$C(\alpha u + \beta v) = \bar{\alpha} Cu + \bar{\beta} Cv, \quad u, v \in \mathcal{H}$$

is called a conjugation if it is norm preserving

$$\|Cu\| = \|u\| \qquad \forall u \in \mathcal{H}$$

and

$$C^2 = 1.$$

Exercise $(Cu, Cv) = (v, u)$. (Hint: use polarization).

Theorem (von Neumann's Theorem)

Let A be a symmetric operator and suppose that $\exists$ a conjugation C with $C : D(A) \to D(A)$ and $AC = CA$ on $D(A)$. Then A has equal deficiency indices and therefore has self-adjoint extensions.

Proof: As $C^2 = 1$ and $CD(A) \subset D(A)$, we have $CD(A) = D(A)$. Suppose $\varphi_+ \in K_+$ and $\psi \in D(A)$. Then for $\psi \in D(A)$,

$$0 = (\varphi_+, (A+i)\psi) = (C\varphi_+, C(A+i)\psi)$$

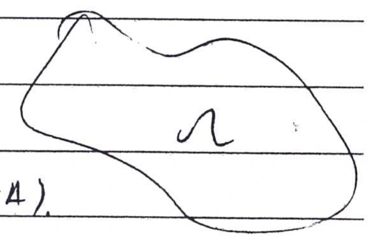
$$= (C\varphi_+, (A-i)C\psi)$$

As C takes $D(A)$ onto $D(A)$. This implies that $C\varphi_+ \in K_-$. Thus C takes $K_+ \to K_-$. Similarly C takes $K_- \to K_+$. Since $C^2 = 1$ we must have $CK_+ = K_-$ and, in particular

$$\dim K_+ = \dim K_- .$$

Examples:

① Let $\mathcal{H} = L^2(\Omega)$ where Ω is a region in $\mathbb{R}^n$. Let $D(-\Delta) = C_0^\infty(\Omega)$



and $-\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$, $f \in D(-\Delta)$.

Let $Cf = \bar{f}$. Then $-\Delta$ is a symmetric operator on $D(-\Delta)$ and C is a conjugation that commutes with $-\Delta$, $(-\Delta)C = C(-\Delta)$ on $D(-\Delta)$.

Thus $-\Delta \restriction C_0^\infty(\Omega)$ has self-adjoint extensions.

(2) In particular for $n=1$ with $\Omega = (0,b)$, $0<b\le\infty$

$-d^2/dx^2 \upharpoonright C_0^\infty(0,b)$ has s.adj. extensions. By contrast, we know that $T = i\frac{d}{dx}$ is a sym. op. on $C_0^\infty(0,\infty)$, but it has no self-adjoint extensions: this is reflected in the fact that the operator T does not commute with complex conjugation.
The fact that $T \upharpoonright C_0^\infty(0,b)$, $b<\infty$, has s.adj extens, shows that commuting with a conjugation, is not a necessary condition.

Example: Let $AC^2[0,1]$ denote the functions $f \in L^2(0,1)$ such that f is abs. cont, f' is abs. cont and $f'' \in L^2(0,1)$

Define

$$D_0 = \{f : f \in AC^2[0,1],\ f(0) = f(1) = 0 = f'(0) = f'(1)\}$$

For a, b real,

$$D_{a,b} = \{f : f \in AC^2(0,1),\ af(0) + f'(0) = 0,\ bf(1) + f'(1) = 0\}$$

$$D_{\infty,\infty} = \{f : f \in AC^2[0,1],\ f(0) = 0 = f(1)\}$$

$$D_{per} = \{f \in AC^2[0,1] : f(1) = f(0),\ f'(1) = f'(0)\}$$

$$D = \{f \in AC^2[0,1]\}$$

and let $T_0, T_{a,b}, T_{\infty,\infty}, T_{per}$ and T be the operators $-d^2/dx^2$ with domains $D_0, D_{a,b}, D_{\infty,\infty}, D_{per}, D$ respectively.

Then

(a) The operators $T_0, T_{a,b}, T_{\infty,\infty}, T_{per}$ and T are closed.

T_0 is symmetric but not s. adj; its adjoint is T

(b) $T_{a,b}, T_{\infty,\infty}$ an T_{per} are self-adjoint extensions of T_0.

$-\infty < a < \infty,\ -\infty < b < \infty$ (There are of course many others!).

Proof: Exercise. Note that once (a) has been established, (b) is easy to verify. Just show in each case that $K_+ = K_- = \{0\}$.

Question: Can we identify each of the extensions $T_{a,b}, T_{\infty,\infty}$ an T_{per} with an isometry $U : K_+ \to K_-$ in Th^m 103.2 ?

We consider, in particular, $T_{\infty,\infty}$.

$K_+ = \{ f \in AC^2[0,1] \;:\; T_0^* f = if \}$

$K_- = \{ f \in AC^2[0,1] : T_0^* f = -if$

Thus $f \in K_+ \iff -f'' = if \iff f = ce^{\lambda x}$ with $-\lambda^2 = i$

i.e., $\lambda = \pm\lambda_+$, $\lambda_+ = e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$

$\times -\lambda_+$

$\times \lambda_+$

Similarly

$f \in K_- \iff -f'' = -if$

$\iff f = ce^{\lambda x}$ with $-\lambda^2 = -i$

$\iff \lambda = \pm\lambda_-$, $\lambda_- = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$

$\cdot \lambda_-$

Note that $\overline{\lambda_+} = \lambda_-$.

$\cdot -\lambda_-$

Set

$$(117.1)\quad \begin{cases} u_+ = e^{\lambda_+ x} + e^{\lambda_+(1-x)} \\ v_+ = e^{\lambda_+ x} - e^{\lambda_+(1-x)} \end{cases}$$

and

$$(117.2)\quad \begin{cases} u_- = e^{\lambda_- x} + e^{\lambda_-(1-x)} \\ v_- = e^{\lambda_- x} - e^{\lambda_-(1-x)} \end{cases}$$

Note that

(118.1) $$u_\pm(x) = u_\pm(1-x) \quad , \quad v_\pm(x) = -v_\pm(1-x)$$

and hence

(118.2) $$\{u_+, v_+\} \quad \text{and} \quad \{u_-, v_-\}$$

provide orthogonal bases for K_+ and K_- respectively.
It is easy to show that if $f \in D(T_u)$, then as in (112.1), $T_u f = -f''$.
We seek a unitary map $U : K_+ \to K_-$ such that

$\mathcal{D}_U = \{\varphi + \varphi_+ + U\varphi_+ : \varphi_+ \in K_+\} = D_{\infty,\infty}$. As

we now show, we can construct U in the form

(118.3) $$U u_+ = \chi_u u_- \quad , \quad U v_+ = \chi_v v_-$$

with $|\chi_u| = |\chi_v| = 1$. Such a U is unitary by (118.2)

and the fact that $u_+ = \bar{u}_-$ and $v_+ = \bar{v}_-$, and hence

$\|u_+\| = \|u_-\|$ and $\|v_+\| = \|v_-\|$. The only issue is to show that $D_u = D_{\infty,\infty}$. For this it is enough to show that $D_u \subset D_{\infty,\infty}$, as s.adjoint operators are maximally symmetric i.e. if $T_u \subset T_{\infty,\infty}$ with $T_{\infty,\infty}$ s.adj, and hence symmetric, then $T_{\infty,\infty} \subset T^*_{\infty,\infty} \subset T_u^* = T_u$, so $T_u = T_{\infty,\infty}$. To show $\mathcal{D}_u \subset D_{\infty,\infty}$ it is enough in turn to show that, for suitable χ_u, χ_v, and $\quad u_+(0) + \chi_u u_-(0) = 0 \quad , \quad u_+(1) + \chi_u u_-(1) = 0$

$\quad\quad v_+(0) + \chi_v v_-(0) = 0 \quad , \quad v_+(1) + \chi_v v_-(1) = 0$

But $u_\pm(1) = u_\pm(0)$ and $v_\pm(1) = -v_\pm(0)$ so it is enough to verify $u_+(0) + \chi_u u_-(0) = (1+e^{\lambda_+}) + \chi_u(1+e^{\lambda_-}) = 0$ and

$$v_+(0) + \chi_v v_-(0) = (1 - e^{\lambda_+}) + \chi_v(1 - e^{\lambda_-}) = 0 \quad \text{and so}$$

(119.0) $\chi_u = -\dfrac{1+e^{\lambda_+}}{1+e^{\lambda_-}}$, $\chi_v = -\dfrac{1-e^{\lambda_+}}{1-e^{\lambda_-}}$. As $\lambda_- = \bar{\lambda}_+$, we clearly have

$$|\chi_u| = |\chi_v| = 1 .$$

This completes the construction of U with the desired properties. $\square$

Exercise Find U for $T_{a,b}$, $-\infty < a, b < \infty$ and T_{per}.

$$\|u_+\|^2 = \|a u_- + b v_-\|^2$$
$$\|v_+\|^2 = \|c u_- + d v_-\|^2$$
$$(u_+, v_+) = (a u_- + b v_- , c u_- + d v_-) .$$

Lecture 10

Another very elegant application of von Neumann's theorem concerns the Hamburger moment problem:

Let ρ be a pos. measure on $\mathbb{R}$ with finite moments and define

(119.1) $$a_n = \int x^n \, d\rho(x) \quad , \quad n = 0, 1, 2, \ldots$$

The #'s a_n are called the moments of the meas. ρ.

The Hamburger moment problem is to determine conditions on a sequence of real #'s $\{a_n\}_{n \geq 0}$ so that $\exists$ a meas. satisfying (119.1)