We conclude from the above that if

$$Tf = f', \qquad D(T) = C_0^\infty$$

then $\qquad D(\bar{T}) = \{ f \in L^2 : f \text{ ac}, \ f' \in L^2 \}$

$$\bar{T} f = f'$$

Lecture 3

Definition 35.1: adjoint operator T^*

Let T be a densely defined operator in $\mathcal{H}$. Let $D(T^*)$ be the set of $\varphi \in \mathcal{H}$ for which $\exists\, \eta \in \mathcal{H}$ s.t.

(35.2) $$(T\psi, \varphi) = (\psi, \eta) \qquad \forall\, \psi \in D(T)$$

Then for $\varphi \in D(T^*)$,

(35.3) $$T^*\varphi = \eta$$

Note that as T is densely defined, η is unique in (35.2) and hence T^* is well-defined.

~~Note: $\varphi \in D(T^*)$ if and only if $(T\psi, \varphi)$ is a bdd. anti- funct. wrt ψ.~~

Note:

(36.1) $\varphi \in D(T^*) \iff |(T\psi, \varphi)| \le c\,\|\psi\| \quad \forall\, \psi \in D(T)$
(why?)

(36.2) $S \subset T \Rightarrow T^* \subset S^*$.

Clearly $0 \in D(T^*)$, but unlike the case of bded operators, $D(T^*)$ may not be dense. In fact, there are examples where $D(T^*) = \{0\}$.

Exple 36.3

Let f be a bounded measurable function such that $f \notin L^2(\mathbb{R})$. Set $D(T) = \{\psi \in L^2(\mathbb{R}) : \int |f(x)\,\psi(x)|\,dx < \infty\}$.

Let $\psi_0 \neq 0$ be some fixed vector in $L^2(\mathbb{R})$. Define

$$T\psi = (f, \psi)\,\psi_0 \equiv \left(\int \overline{f(x)}\,\psi(x)\,dx\right)\psi_0, \qquad \psi \in D(T).$$

As $D(T)$ certainly contains all L^2 functions with compact support, we see that T is densely defined.

Now suppose $\varphi \in D(T^*)$. Then for some η

$(= T^*\varphi) \in L^2$

$$(T\psi, \varphi) = (\psi, \eta) \qquad \forall \psi \in D(T)$$

ie $\left(\int f(x)\overline{\psi(x)}\,dx\right)(\psi_0, \varphi) = (\psi, \eta)$.

ie $\int [(\psi_0, \varphi) f(x) - \eta(x)]\overline{\psi(x)}\,dx = 0$.

As ψ can be any L^2 func. with compact supp we must have $\eta(x) = (\psi_0, \varphi) f(x)$

But RHS $\in L^2 \iff (\psi_0, \varphi) = 0$. Thus $D(T^*)$ is $\perp$ to ψ_0 and hence cannot be dense. Note also that $T^*\varphi = \eta = 0$ for $\varphi \in D(T^*)$.

<u>Exercise</u>: Extend the above construction to show $\exists$ T densely defined st $D(T^*) = \{0\}$

Observe that T in 36.3 is not closable.

Indeed as $f \notin L^2$, $\exists$ compact subsets S_n of L^2

such that
$$\int_{S_n} |f(x)|^2 dx = n.$$

Let $\psi_n = \frac{f \chi_{S_n}}{n}$, $n \geq 1$. Clearly $\psi_n \in D(T)$ and

$$\|\psi_n\|^2 = \frac{1}{n^2} \int_{S_n} |f|^2 = \frac{1}{n} \to 0 \quad \text{i.e. } \psi_n \to 0$$

as $n \to \infty$. But $T\psi_n = (f, \psi_n)\psi_0 = \left(\frac{1}{n} \int_{S_n} |f^2|\right) \psi_0 = \psi_0$

So $\psi_n \to 0$ but $T\psi_n \not\to 0$.

It turns out that there is an intimate relationship between the closability of an operator and the dense definition of its adjoint.

Note that if T^* is densely defined, then we may define $T^{**} = (T^*)^*$.

<u>Th^m 38.1</u> Let T be a densely defined operator in $\mathcal{H}$. Then

(a) T^* is closed

(b) T is closable $\Leftrightarrow$ T^* is densely defined, in which case $\overline{T} = T^{**}$

(c) If T is closable, then $(\overline{T})^* = T^*$.

<u>Proof</u> (a): Suppose $D(T^*) \ni \varphi_n \to f$
$T^*\varphi_n \to g$.

Now $\forall \psi \in D(T)$, $(T\psi, \varphi_n) = (\psi, T^*\varphi_n)$. Letting $n \to \infty$ we obtain $(T\psi, f) = (\psi, g)$ $\forall \psi \in D(T)$.

Hence $f \in D(T^*)$ and $T^*f = g$. Thus T^* is closed.

(b) Suppose $D(T^*)$ is dense and $\phi \in D(T^*)$. Then

(39.1) $$(T\psi, \phi) = (\psi, T^*\phi) \quad \forall \psi \in D(T)$$

Now suppose $D(T) \ni \psi_n \to 0$
$T\psi_n \to g$

Then from (39.1), $(g, \phi) = (0, T^*\phi) = 0$.

But as $D(T^*)$ is dense, we conclude $g = 0$. Thus T is closable.

The proof of the converse is more involved. Let

$\Gamma(T)$ = graph of $T = \{\langle x, Tx\rangle : x \in D(T)\} \subset \mathcal{H} \times \mathcal{H}$.

$\mathcal{H} \times \mathcal{H}$ is a Hilbert space with inner product $(\langle x, y\rangle, \langle x', y'\rangle)$
$$(\langle x, y\rangle, \langle x', y'\rangle) = (x, x') + (y, y').$$

Now

$$\langle f, g\rangle \perp \Gamma(T) \qquad \text{in } \mathcal{H}\times\mathcal{H}$$
$$\Leftrightarrow \quad (f,\psi) + (g, T\psi) = 0 \qquad \forall \psi \in D(T)$$
$$\Leftrightarrow \quad (g, T\psi) = (-f, \psi) \qquad \forall \psi \in D(T)$$
$$\Leftrightarrow \quad g \in D(T^*) \text{ and } T^* g = -f$$

(40.1) Hence, $\Gamma(T)^{\perp} = \{\langle -T^* g, g\rangle : g \in D(T^*)\}$

Now (exercise):

T is closable $\Leftrightarrow$ $\overline{\Gamma(T)}$ is a graph and $\overline{\Gamma(T)} = \Gamma(\overline{T})$

Clearly T is closed $\Leftrightarrow$ $\Gamma(T)$ is closed

Thus if T is closable, then from (40.1)

$$\mathcal{H}\times\mathcal{H} = \overline{\Gamma(T)} \oplus (\overline{\Gamma(T)})^{\perp} = \Gamma(\overline{T}) \oplus (\Gamma(T))^{\perp}.$$

(Here we use $(\overline{\Gamma(T)})^{\perp} = (\Gamma(T))^{\perp}$.)

Thus ~~for~~ any $\langle f, g\rangle \in \mathcal{H}\times\mathcal{H}$ can be written uniquely as

$$\langle f, g\rangle = \langle \psi, \overline{T}\psi\rangle + \langle -T^*\phi, \phi\rangle, \quad \psi \in D(\overline{T}),\ \phi \in D(T^*)$$

If $D(T^*)$ is <u>not</u> dense, then $\exists\, \varphi \neq 0$, $\varphi \perp D(T^*)$.

Then

$$\langle 0, \varphi\rangle = \langle \psi_0, \overline{T}\psi_0\rangle + \langle -T^*\phi_0, \phi_0\rangle \text{ for suitable}$$

$\psi_0 \in D(\bar{T})$, $\phi_0 \in D(T^*)$

i.e.

(41.1) $0 = \psi_0 - T^* \phi_0$

(41.2) $\varphi = \bar{T}\psi_0 + \phi_0$

As $(\varphi, \phi_0) = 0$, we have from (41.2)

$$\|\phi_0\|^2 + (\bar{T}\psi_0, \phi_0) = 0$$

But by part (c) (see below), $\bar{T}^* = T^*$, and so

$$\|\phi_0\|^2 + (\psi_0, T^*\phi_0) = 0$$

By (41.1) we then obtain

$$\|\phi_0\|^2 + \|\psi_0\|^2 = 0$$

$\therefore$ $\phi_0 = \psi_0 = 0 \Rightarrow \varphi = 0$, which is a contradiction

Thus T^* is densely defined.

For T closable, $\boxed{\Gamma(\bar{T}) = \overline{\Gamma(T)} \text{ is closed,}}$ and from (40.1) (recall $X = X^{\perp\perp}$ for any closed set $X \subset \mathcal{H}$).

$$\Gamma(\bar{T}) = \Gamma(\bar{T})^{\perp\perp}$$
$$= (\Gamma(T))^{\perp\perp}$$
$$= \{\langle -T^*g, g\rangle : g \in D(T^*)\}^{\perp}$$

but by the proof of (40.1), this is just $\Gamma(T^{**})$

Thus $\bar{T} = T^{**}$

Finally we prove (c). Suppose T is closable. We have that $\varphi \in D(T^*) \Leftrightarrow (T\psi, \varphi) = (\psi, T^*\varphi)\ \forall \psi \in D(T)$

But then closing T, we obtain $(\bar{T}\psi, \varphi) = (\psi, T^*\varphi)$

$\forall \varphi \in D(\bar{T})$. Hence $T^* \subset (\bar{T})^*$. Conversely, as

$T \subset \bar{T}$, we always have $(\bar{T})^* \subset T^*$. $\therefore\ \bar{T}^* = T^*$.

<u>Definition 42.1</u> (spectrum & resolvent set)

Let T be a (densely defined) closed operator in $\mathcal{H}$.

A complex # λ is in the <u>resolvent set</u> $\rho(T)$ of T if

$\lambda - T = \lambda \mathbb{1} - T$ is a bijection of $D(T)$ <u>onto</u> $\mathcal{H}$.

If $\lambda \in \rho(T)$, $R_\lambda(T) = (\lambda - T)^{-1} : \mathcal{H} \to D(T)$ is called

the <u>resolvent</u> of T at λ. Have $R_\lambda(T)(\lambda - T) = \mathbb{1}_{D(T)}$, $(\lambda - T) R_\lambda(T) = \mathbb{1}_{\mathcal{H}}$.

<u>Exercise</u> If T is closed on $D(T)$, then $\lambda - T$ is closed on $D(T)$ $\forall \lambda \in \mathbb{C}$

A very important fact is that if $\lambda \in \rho(T)$, then

$R_\lambda(T)$ is a bounded operator. This fact lies at

the heart of the analytic viability of closed operators.

As $R_\lambda(T)$ is everywhere defined, it is sufficient by the Closed Graph Theorem to show that if

$$f_n \to f \quad \text{and} \quad R_\lambda(T) f_n \to g$$

$\cancel{\Rightarrow (f \in Dom(R_\lambda(T))) \text{ and } (R_\lambda(T) f = g.)}$

Then $R_\lambda(T) f = g$. But $R_\lambda(T) f_n \in Dom(T)$
$= D(\lambda - T)$

and $(\lambda - T)(R_\lambda(T) f_n) = f_n \to f$

Also $R_\lambda(T) f_n \to g$

Hence as $\lambda - T$ is closed on $D(T)$, $g \in D(\lambda - T)$

and $(\lambda - T) g = f$ ie $g = R_\lambda(T) f$, which is what we wanted to prove.

Th^m 43.1

Let T be closed in $\mathcal{H}$. Then $\rho(T)$ is an open subset of $\mathbb{C}$ on which $R_\lambda(T)$ is an analytic operator valued function. Furthermore

$$\{R_\lambda(T) : \lambda \in \rho(T)\}$$

is a commuting family of bounded operators satisfying

(44.1) $$R_\lambda(T) - R_\mu(T) = (\mu-\lambda)\, R_\mu(T) R_\lambda(T).$$

Proof: Exercise.

We define

$$\sigma(T) = \text{spectrum of } T = \mathbb{C} \setminus \rho(T)$$

Clearly $\sigma(T)$ is a closed subset of $\mathbb{C}$. The spectrum $\sigma(T)$ can be broken up in different ways into point spectrum, etc., as mentioned in Lecture 1.

Exercise: Let Ω be a closed set in $\mathbb{C}$. Show that Ω is the spectrum of some operator in $L^2(\mathbb{R}^2)$ (hint: Show that Ω has a countable dense subset).

The spectrum of an operator is a subtle matter as we see from the following 2 exples:

Let $AC[0,1] = \{f \in L^2(0,1) : f$ is absolutely cont. on $[0,1]$, $f'(x) \in L^2(0,1)\}$.

Let

$$D(T_1) = \{\varphi : \varphi \in AC[0,1]\}$$
$$D(T_2) = \{\varphi : \varphi \in AC[0,1],\ \varphi(0) = 0\}.$$

and let $T_j f = i f'(x)$ if $f \in D(T_j)$, $j = 1, 2$.

A similar calculation to that given above shows that both T_1 and T_2 are closed operators. However

(45.1) $\sigma(T_1) = \mathbb{C}$

(45.2) $\sigma(T_2) = \emptyset$

Proof: To see that $\sigma(T_1) = \mathbb{C}$ simply observe that $0 \neq e^{-i\lambda x} \in D(T_1)$ for any $\lambda \in \mathbb{C}$ as

$$(\lambda - T)\, e^{-i\lambda x} = [\lambda + i(i\lambda)]e^{-i\lambda x} = 0$$

so that $\lambda - T$ is not 1-1 for any λ. As for T_2, suppose that $\lambda \in \mathbb{C}$ and $g \in L^2(0,1)$ are given and we try to solve the eqtn

(45.3) $$(\lambda - T_2) f = g$$

for $f \in D(T_2)$. Then necessarily $\lambda f - i f' = g$

i.e. $-i \frac{d}{dx}(e^{i\lambda x} f) = g \Rightarrow$

$$-i\, e^{i\lambda x} f(x) = -i\, e^{i\lambda x} f(x)\Big|_{x=0} + \int_0^x e^{i\lambda t} g(t)\, dt$$

$$= \int_0^x e^{i\lambda t} g(t)\, dt$$

$$\therefore \quad f(x) = i \int_0^x e^{i\lambda(t-x)} g(t)\, dt$$

(46.1) $$\equiv (S_\lambda g)(x).$$

So we take (46.1) as our starting point. Observe that given λ, g, $f = S_\lambda g(x) \in AC[0,1]$ and $f(0) = 0$, so $f \in \mathcal{D}(T_1)$. Also

$$(\lambda - T_2) f = \left(\lambda - i\frac{d}{dx}\right) i \int_0^x e^{i\lambda(t-x)} g(t)\, dt$$

$$= \lambda i \int_0^x e^{i\lambda(t-x)} g(t)\, dt + e^{i\lambda(x-x)} g(x) - i\lambda \int_0^x e^{i\lambda(t-x)} g(t)\, dt$$

$$= g(x)$$

Thus $\lambda - T_2$ is onto. Also, for $f \in D(T_2)$ and $(\lambda - T_2) f = 0$, then, as above $-i \frac{d}{dx}\left(e^{+i\lambda} f\right) = 0$ and so $e^{i\lambda x} f(x) = \text{const} = c$. But $f(0) = 0$ and so $c = 0$. Thus $f(x) \equiv 0$ and so T_2 is 1-1. It follows that $\rho(T) = \mathbb{C}$.

Examples (45.1) (45.2) are in sharp contrast with the situation for bounded operators and self-adjoint operators (see - below).
Firstly if T is bounded then (exercise), $\sigma(T)$ is a <u>bounded</u>, <u>non-empty</u> set. Secondly, if T is self-adjoint (bded or unbded) then

$$\phi \neq \sigma(T) \subset \mathbb{R}.$$

<u>Remark</u>

By general principles noted above, S_λ is bded from $L^2 \to L^2$: it is of interest to check this directly (exercise!)

We now formally distinguish between <u>symmetric</u> <u>operators</u> and <u>self-adjoint</u> operators

<u>Defn</u>: A densely defined operator T on $\mathcal{H}$ is called

symmetric (or Hermitian) if

(48.1) $T \subset T^*$

i.e. if $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi \quad \forall \varphi \in D(T)$

Equivalently

(48.2) $(T\varphi, \psi) = (\varphi, T\psi) \quad \forall \varphi, \psi \in D(T)$.

Definition:

T is called self-adjoint if $T = T^*$ i.e.

T is symmetric and $D(T) = D(T^*)$.

Clearly if T is self-adjoint, then T is closed.

Remarks:

(48.3) If T is is symmetric and bounded, then $\bar{T}$ is s.adj.

(48.4) If T is symmetric, then by (48.1) $D(T^*)$ is dense. Hence T is automatically closable.

As T^* is always closed by Th^m 38.1(a), then if T is symmetric, T^* is a closed extension of T^* and we have

(49.1) $$T \subset \bar{T} = T^{**} \subset T^*$$

In particular for closed sym. operators

(49.2) $$T = T^{**} \subset T^*$$

and for s.adj operators

(49.3) $$T = T^{**} = \bar{T} = T^*$$

Thus

(49.4) a closed sym. op T is s.adj. if and only if T^* is symmetric

Exple: Consider

$$T_3 f = i f'(x)$$

with domain in $L^2(0,1)$

$$D(T_3) = \{ f \in L^2(0,1) : f \in AC[0,1], \; f(0) = f(1) \} = 0 \}$$

As before, we see that T_3 is a closed operator.

It is also symmetric: Indeed for $f, g \in D(T_3)$,

$(f, T_3 g) = \int_0^1 \bar{f}\, i g' = \int \overline{i f'}\, g = (T f_3, g)$

Here we use integration by parts which is valid because $AC[0,1]$ is an algebra (check this!) and f and g vanish at 0 and at 1, so there are no bdry terms.

Claim:

Let

$$D(S) = \{ f \in L^2 : f \in AC[0,1] \}$$

$$Sf = i f' \quad , \quad f \in D(S)$$

Then $S = T_3^*$.

Indeed for $f \in D(S)$ and $g \in D(T_3)$, then

(50.1) $\quad (f, T_3 g) = \;=\; \int_0^1 \bar{f}\, i g' \;=\; (if', g) = (Sf, g)$

Again integ. by parts is valid and as $f(x)$ is cont. on $[0,1]$ (why?) and $g(0) = g(1) = 0$, again there are no bdry terms. Thus by (50.1)

$f \in D(T_3^*)$ and $T_3^* f = Sf$ $\therefore$ $S \subset T_3^*$

Conversely, suppose $f \in D(T_3^*)$ and $g \in D(T_3)$.

Then $(f, T_3 g) = \int_0^1 \bar{f}\, i g' = \int_0^1 \bar{h}\, g$ for some $h \in L^2$,

$h = T_3^* f$. For $H(x) \equiv \int_0^x h(t)\,dt$, we obtain

as on p33 that (the bdry terms drop out as $g(0) = g(1) = 0$)

$$\int_0^1 \bar{f}\, i g' = - \int_0^1 \overline{H(x)}\, g'(x)$$

and so $\int_0^1 \overline{if - H}\; g'(x)\,dx$, $\forall g \in D(T_3)$

and we conclude as before that

$$\begin{aligned} if(x) &= H(x) + c \\ &= \int_0^x h(t)\,dt + c . \end{aligned}$$

Thus $f \in AC[0,1]$ and $if' = h = T_3^* f$

This shows that $T_3^* \subset S$, and hence establishes the claim.

Thus $T_3^* = S = T_1$ (see p44).