$$= \left(\sum_{N < n^2 \le N'} (\Phi_n, f)\Phi_n , \sum_{N < n^2 \le N'} n^2 (\Phi_n, f)\Phi_n \left(\tfrac{\pi}{2}\right)^2 \right)$$

$$= \left(\tfrac{\pi}{2}\right)^2 \sum_{N < n^2 \le N'} |(\Phi_n, f)|^2 n^2 \quad \to 0 \quad \text{as } N' > N \to \infty,$$

by (177.0). Thus $\{\nabla f_N\}$ is Cauchy and so (for some $h \in (L^2(\Omega))^m$)
$\nabla f_N \to h$ in $(L^2(\Omega))^m$ and as

$$(\nabla f_N, \phi) = -(f_N, \nabla\phi)$$

$\forall \phi \in C_0^\infty$, we have $(h, \phi) = (f, \nabla\phi)$

$\forall \phi \in C_0^\infty$. Thus $h = \nabla f$. This shows that
if $f \in H^1(\Omega)$, then
$\exists f_N \in D(B)$ st $f_N \to f$ and $\nabla f_N \to \nabla f$

Taking $\varphi = f_N$ in (175.1) and letting $N \to \infty$,

we conclude that (175.1) holds $\forall f = \varphi \in H^1(\Omega)$

as desired. Thus $-\Delta_N^\Omega = \overline{B}$.

Lecture 14 On a cube $(-a, a)^m$ the eigenvectors and eigenvalues

of $-\Delta_D$ are labelled by m-tuples $n = (n_1, \dots, n_m)$

with $n_i \ge 1$, $i = 1, \dots, m$ and are given by

(178.1) $$\Phi_{n;a}(x) = a^{-m/2} \prod_{i=1}^{m} \varphi_{n_i}(x_i/a)$$

where

$$\varphi_k(x) = \cos(k\pi x/2) \quad , k = 1, 3, 5, \ldots$$
$$\varphi_k(x) = \sin(k\pi x/2) \quad , k = 2, 4, 6, \ldots$$

corresponding to eigenvalues

(179.0) $$E_n(a) = \left(\frac{\pi}{2a}\right)^2 \sum_{i=1}^{m} n_i^2$$

resp. For $-\Delta_N$ in $(-a, a)^m$, the eigenvectors are labelled by m-tuples $n = (n_1, \ldots, n_m)$, $n_i \geq 0$, and are given by

(179.1) $$\Psi_{n;a}(x) = a^{-m/2} \prod_{i=1}^{m} \psi_{n_i}(x_i/a)$$

where (cf (176.21))

$$\psi_k(x) = \sin(k\pi x/2) \quad , k = 1, 3, 5, \ldots$$
$$\psi_k(x) = \cos(k\pi x/2) \quad , k = 2, 4, 6, \ldots$$
$$\psi_k(x) = \frac{1}{\sqrt{2}} \quad , k = 0$$

with eigenvalues again given by (179.0)

Because of this explicit listing one can prove Weyl's law directly for cubes.

Proposition 179.2

Let $N_D(a, \lambda)$ (resp. $N_N(a, \lambda)$) denote the dimension of the spectral projection $P_{[0,\lambda)}$ for $-\Delta_D$ (resp. $-\Delta_N$)

on $(-a,a)^m$. Then for all a, λ, we have.

(180.1) $$\left| N_D(a,\lambda) - \tau_m \lambda^{m/2} \left(\frac{2a}{2\pi}\right)^m \right| \le C\left(1 + (a^2\lambda)^{\frac{m-1}{2}}\right)$$

and

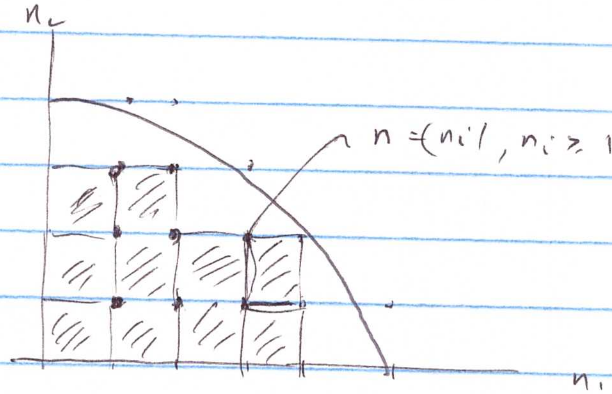
(180.2) $$\left| N_N(a,\lambda) - \tau_m \lambda^{m/2} \left(\frac{2a}{2\pi}\right)^m \right| \le C\left(1 + (a^2\lambda)^{\frac{m-1}{2}}\right)$$

where τ_m is the vol. of the unit ball in $\mathbb{R}^m$ and C is a suitable constant.

<u>Proof:</u> We begin with a <u>remark</u>. On account of (179.0), (178.1), (179.1), N_D (resp. N_N) is the number of points $n = (n_1, \ldots, n_m)$, $n_i \ge 1$, (resp, $n = (n_1, \ldots, n_m)$, $n_i \ge 0$) within the "octant" of ball of radius $\left(\frac{2a}{\pi}\right)\lambda^{1/2}$. For large λ this should be approximately the volume of the "octant" of this sphere i.e. $2^{-m}\tau_m \left(\frac{2a}{\pi}\lambda^{\frac{1}{2}}\right)^m$, and the error term should be a "surface term" i.e. order $(a^2\lambda)^{\frac{m-1}{2}}$.

Let S_λ denote the ball of radius $\left(\frac{2a}{\pi}\right)\lambda^{\frac{1}{2}}$. For each

$n = (n_i)$ in S_λ, $n_i \geq 1$, the unit cube $\{x : n_i - 1 \leq x_i \leq n_i\}$ is contained in the upper octant S_λ,



so

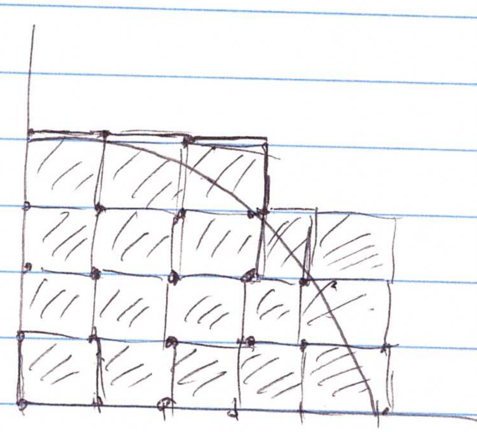
(181.1) $$N_D(\lambda) \leq \frac{1}{2^m} \tau_m \left(\frac{2a}{\pi} \lambda^{\frac{1}{2}}\right)^m = \tau_m \left(\frac{2a}{2\pi}\right)^m \lambda^{m/2}$$

~~On the other hand, if $n^2 < \left(\frac{2a}{\pi}\right)^2 \lambda$, $n_i \geq 0$ but $(n')^2 \geq \left(\frac{2a}{2\pi}\right)^2 \lambda$, where for some $k \in \{1, \ldots, m\}$ $n'_i = n_i$ for $i \neq k$, $n'_i = n_i + 1$ for $i = k$.~~

On the other hand, suppose $x = (x_1, \ldots, x_m)$, $x_i \geq 0$, has norm $x^2 < \left(\frac{2a}{\pi} \lambda^{1/2}\right)^2$. Then x lies in the cube

$Q_n = [n_1, n_1+1] \times [n_2, n_2+1] \times \cdots \times [n_m, n_m+1]$ for some $n = (n_1, \ldots, n_m)$, $n_i \geq 0$. But $x^2 \geq n^2$ and so $\left(\frac{2a}{\pi}\lambda^{\frac{1}{2}}\right)^2 > n^2$ and so $S_\lambda = \{x : x^2 < \left(\frac{2a}{\pi} \lambda^{\frac{1}{2}}\right)^2\} \subset \bigcup_{n^2 < \left(\frac{2a}{\pi}\lambda^{1/2}\right)^2} Q_n$. Thus

(182.1) $$N_N(\lambda) \geq \tau_m \left(\frac{2a}{2\pi}\right)^m \lambda^{m/2}$$

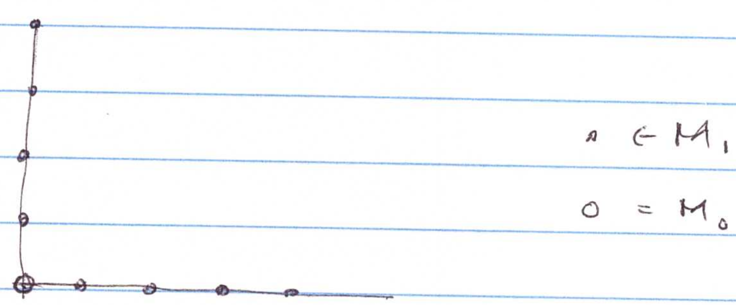


Now it is clear that

$$N_N(\lambda) - N_D(\lambda) = \bigcup_{k=0}^{m-1} M_k(\lambda)$$

where

$$M_k(\lambda) = \{ n = (n_1, \dots, n_m) \in S_\lambda,\ n_i \geq 0 : \text{exactly } k \text{ of the } n_i \text{ are non-zero} \}$$



As in our bound on N_D, $\# M_k(\lambda) \leq \binom{m}{k} \tau_k \left(\frac{2a}{2\pi}\right)^k \lambda^{k/2}$

(182.2) Therefore $$\sum_{k=0}^{m-1} \# M_k(\lambda) \leq \text{const} \left(\sum_{k=0}^{m-1} (a^2\lambda)^{k/2} \binom{m}{k} \right) \leq C\left(1 + (a^2\lambda)^{(m-1)/2}\right).$$

Thus from (181.1) (182.1) (182.2)

$$
\begin{aligned}
0 &\le N_N(\lambda) - \tau_m \left(\tfrac{2a}{2\pi}\right)^m \lambda^{m/2} \\
&\le N_N(\lambda) - N_D(\lambda) \\
&\le C\left(1 + (a^2\lambda)^{\frac{m-1}{2}}\right)
\end{aligned}
$$

and similarly.

$$
\begin{aligned}
0 &\ge N_D(\lambda) - \tau_m \left(\tfrac{2a}{2\pi}\right)^m \lambda^{m/2} \\
&\ge N_D(\lambda) - N_N(\lambda) \\
&\ge -C\left(1 + (a^2\lambda)^{\frac{m-1}{2}}\right)
\end{aligned}
$$

This proves (180.1) and (180.2) $\square$

The following Proposition is easy to verify and is left as an exercise.

Proposition (183.1)

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let A_1 and A_2 be s.adj. operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $A = A_1 \oplus A_2$ be the operator in $\mathcal{H}$ with domain $\{\varphi \oplus \psi : \varphi \in D(A_1), \psi \in D(A_2)\}$.

Then

(1) $A_1 \oplus A_2$ is s.adj.

(2) If D_1 is a core for A_1 and D_2 is a core for A_2, then $D_1 \oplus D_2 = \{\varphi \oplus \psi : \varphi \in D_1, \psi \in D_2\}$ is a core for $A_1 \oplus A_2$

(3) $Q(A_1 \oplus A_2) = Q(A_1) \oplus Q(A_2)$ and if $\varphi \oplus \psi \in Q(A_1) \oplus Q(A_2)$ then

$(\varphi \oplus \psi, (A_1 \oplus A_2)\varphi \oplus \psi) = (\varphi, A_1 \varphi) + (\psi, A_2 \psi)$

(4) For any Borel set $\Omega \subset \mathbb{R}$

$$P_\Omega(A_1 \oplus A_2) = P_\Omega(A_1) \oplus P_\Omega(A_2).$$

(5) If $N(\lambda; A) = \dim P_\lambda(A)$ then

$$N(\lambda, A_1 \oplus A_2) = N(\lambda, A_1) + N(\lambda, A_2)$$

Proposition (184.1)

Let Ω_1 and Ω_2 be disjoint open sets so that

$L^2(\Omega_1 \cup \Omega_2) = L^2(\Omega_1) \oplus L^2(\Omega_2)$. Under this decomposition

(184.2) $$-\Delta_D^{\Omega_1 \cup \Omega_2} = -\Delta_D^{\Omega_1} \oplus -\Delta_D^{\Omega_2}$$

(184.3) $$-\Delta_N^{\Omega_1 \cup \Omega_2} = -\Delta_N^{\Omega_1} \oplus -\Delta_N^{\Omega_2}$$

Proof: We consider the Dirichlet case: the Neumann case is similar. Given $f \in C_0^\infty(\Omega_1 \cup \Omega_2)$, let

$f_i = f \restriction \Omega_i$, $i = 1, 2$. Then $f_i \in C_0^\infty(\Omega_i)$ and

clearly for $f, g \in C_0^\infty(\Omega_1 \cup \Omega_2)$

$$\int_{\Omega_1 \cup \Omega_2} \overline{\nabla f} \cdot \nabla g \, d^m x = \int_{\Omega_1} \overline{\nabla f_1} \cdot \nabla g_1 \, d^m x + \int_{\Omega_2} \overline{\nabla f_2} \cdot \nabla g_2 \, d^m x$$

and the relation extends to the closure of the quadratic forms. The quadratic forms are equal and the result follows. $\square$.

Corollary 185.1 Let $N_D(\Omega, \lambda)$ (resp. $N_N(\Omega, \lambda)$) be the dimensions of the spectral proj. $P_{[0,\lambda)}$ for $-\Delta_D^\Omega$ (resp $-\Delta_N^\Omega$). Then if $\Omega_1, \ldots, \Omega_k$ are disjoint, then

(185.2) $$N_D\left(\bigcup_{i=1}^k \Omega_i, \lambda\right) = \sum_{i=1}^k N_D(\Omega_i, \lambda)$$

(185.3) $$N_N\left(\bigcup_{i=1}^k \Omega_i, \lambda\right) = \sum_{i=1}^k N_N(\Omega_i, \lambda).$$

We need to extend the definition of $A \le B$ to the unbounded situation:

Defn (186.1) Let A and B be s.adj. ops that are non-negative where A is defined on a dense subspace of a Hilbert space $\mathcal{H}$ and B is defined on a dense subspace of a Hilbert subspace $\mathcal{H}_1 \subset \mathcal{H}$. We write $0 \le A \le B$ if and only if

(i) $Q(B) \subset Q(A)$

(ii) For any $\varphi \in Q(B) \subset Q(A)$

$$0 \le (\varphi, A\varphi) \le (\varphi, B\varphi).$$

Example $q_N = \int_0^1 |f'|^2$, $Q(q_N) = \{f \in AC[0,1] ; f' \in L^2\}$

$q_D = \int_0^1 |f'|^2$, $Q(q_D) = \{f \in AC[0,1] : f' \in L^2$
$f(0) = f(1) = 0\}$

Then $0 \le -(d^2/dx^2)_N \le -(d^2/dx^2)_D$

By min-max, we immediately have

Lemma 187.1

If $0 \le A \le B$, then:

(a) $\dim P_{[0,\lambda)}(A) \ge \dim P_{[0,\lambda)}(B)$ for all $\lambda > 0$

(b) $\mu_n(A) \le \mu_n(B)$ $\forall n$, where μ_n is given by min-max.

Proof: Let $Q(B) \subset \mathcal{H}_1 \subset \mathcal{H}$. Let $\pi : \mathcal{H} \to \mathcal{H}_1$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1$.

Let $\psi_1, \ldots, \psi_{n-1} \in \mathcal{H}$. Then as $Q(A) \supset Q(B)$,

$$\min_{\substack{\varphi \in Q(A) \\ \varphi \perp (\psi_1, \ldots, \psi_{n-1}) \\ \|\varphi\| = 1}} (\varphi, A\varphi) \le \min_{\substack{\varphi \in Q(B) \\ \varphi \perp (\psi_1, \ldots, \psi_{n-1}) \\ \|\varphi\| = 1}} (\varphi, A\varphi)$$

$$= \min_{\substack{\varphi \in Q(B) \\ \varphi \perp (\pi\psi_1, \ldots, \pi\psi_{n-1}) \\ \|\varphi\| = 1}} (\varphi, A\varphi)$$

$$\le \min_{\substack{\varphi \in Q(B) \\ \varphi \perp (\pi\psi_1, \ldots, \pi\psi_{n-1}) \\ \|\varphi\| = 1}} (\varphi, B\varphi)$$

$$\le \mu_n(B).$$

and so $\mu_n(A) \le \mu_n(B)$.

Now suppose $\dim P_{[0,\lambda)}(A) < \dim P_{[0,\lambda)}(B)$ for some λ. Then, in particular, $\dim P_{[0,\lambda)}(A) < \infty$, say $\dim P_{[0,\lambda)}(A) = n$ for some $n < \infty$. Then by min-max, as $\dim P_{[0,\lambda)}(A) < n+1$, we have $\lambda < \mu_{n+1}(A)$. However, $\dim P_{[0,\lambda)}(B) \ge n+1$ and so $\lambda > \mu_{n+1}(B)$. But $\mu_{n+1}(A) \le \mu_{n+1}(B)$ and so $\lambda > \mu_{n+1}(B) \ge \mu_{n+1}(A) > \lambda$ which is a contradiction. Hence $\dim P_{[0,\lambda)}(A) \ge \dim P_{[0,\lambda)}(B)$. □

<u>Proposition 188.1</u>

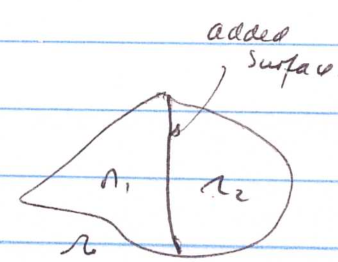
(a) If $\Omega \subset \Omega'$, then $0 \le -\Delta_D^{\Omega'} \le -\Delta_D^{\Omega}$

(b) For any Ω, $0 \le -\Delta_N^{\Omega} \le -\Delta_D^{\Omega}$

(c) Let Ω_1, Ω_2 be disjoint open subsets of an open set Ω so that

$$(\overline{\Omega_1 \cup \Omega_2})^{int} = \Omega.$$

and $\Omega \setminus (\Omega_1 \cup \Omega_2)$ has measure 0



Then

(189.1) $$0 \leq -\Delta_D^{\Omega} \leq -\Delta_D^{\Omega_1 \cup \Omega_2}$$

(189.2) $$0 \leq -\Delta_N^{\Omega_1 \cup \Omega_2} \leq -\Delta_N^{\Omega}$$

Proof (a) This statement is to be interpreted in the sense that $f \in L^2(\Omega)$ is viewed as an element of $L^2(\Omega')$ by setting it equal to zero in $\Omega' \setminus \Omega$. With this definition

$$C_0^\infty(\Omega) \subset C_0^\infty(\Omega') \quad \text{and} \quad -\Delta_D^{\Omega'} \upharpoonright C_0^\infty(\Omega) \times C_0^\infty(\Omega)$$

$= -\Delta_D^{\Omega}$ as quadratic forms and so (a) follows

(b) Since $C_0^\infty(\Omega) \subset H^1(\Omega)$, this is immediate.

(c) Clearly $C_0^\infty(\Omega_1 \cup \Omega_2) \subset C_0^\infty(\Omega)$. Now argue as in (a) to prove (189.1). On the other hand, if $f \in H^1(\Omega)$, its restriction to $\Omega_1 \cup \Omega_2$ is clearly in $H^1(\Omega_1) \oplus H^1(\Omega_2)$. Moreover since $\Omega \setminus \Omega_1 \cup \Omega_2$ has measure zero,

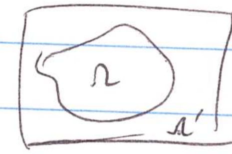
$$\int_\Omega |\nabla f|^2 dx = \int_{\Omega_1 \cup \Omega_2} |\nabla f|^2 dx. \quad \square$$

Thus adding in a Dirichlet boundary term pushes the eigenvalues up: but adding in a Neumann boundary condition, pushes the eigenvalues down.

<u>Corollary 190.1</u>

Let Ω be an open, bounded set in $\mathbb{R}^m$. Let $-\Delta_D^\Omega$ denote the Dirichlet operator in Ω. Then $-\Delta_D^\Omega$ has purely discrete spectrum $\lambda_n \to \infty$.

<u>Proof</u> Enclose Ω by a cube Ω'



Then by Prop 188.1 (a), $0 \le -\Delta_D^{\Omega'} \le -\Delta_D^{\Omega}$ and hence by Lemma 187.1, $\mu_n(-\Delta_D^{\Omega'}) \le \mu_n(-\Delta_D^{\Omega})$

But we know that $-\Delta_D^{\Omega'}$ has purely discrete spectrum with eigs $\lambda_n(\Omega') \to \infty$. But then by min-max

$$\mu_n(-\Delta_D^{\Omega}) \ge \mu_n(-\Delta_D^{\Omega'}) = \lambda_n(\Omega') \to \infty$$

and so $-\Delta_D^\Omega$ has no ess. spec., again by min-max. $\square$

Notice that the result above makes <u>no</u> assumptions on $\partial\Omega$.

Definition 191.1

A <u>standard 2^{-n} cube in $\mathbb{R}^m$</u> is a cube of the form
$$\left[a_1/2^n, (a_1+1)/2^n\right) \times \cdots \times \left[\frac{a_m}{2^n}, (a_m+1)/2^n\right)$$
in $\mathbb{R}^n$ with $a_1, \ldots, a_m$ integers. Given a set $\Omega \subset \mathbb{R}^m$, we let $W_n^-(\Omega)$ be the volume of those standard 2^{-n} cubes contained in Ω and $W_n^+(\Omega)$ be the volume of standard 2^{-n} cubes that intersect Ω. Thus if Ω is Lebesgue measurable,

(191.1) $$W_n^-(\Omega) \le W_{n+1}^-(\Omega) \le \mu(\Omega) \le W_{n+1}^+(\Omega) \le W_n^+(\Omega)$$

The limit $\lim W_n^-(\Omega) = W_\infty^-(\Omega)$ (resp. $\lim W_n^+(\Omega) = W_\infty^+(\Omega)$) is called the <u>inner</u> (resp. <u>outer</u>) <u>Jordan content</u> of Ω. If $W_\infty^+(\Omega) = W_\infty^-(\Omega)$, then we say that Ω is a <u>contented set</u> and $W(\Omega) = W^\pm(\Omega)$ is called its <u>content</u>.

Note that by (191.1), if Ω is contented and Lebesgue measurable, then $W(\Omega) = \mu(\Omega)$. One can also prove that a contented set is Lebesgue measurable (<u>exercise</u>).

Finally we are able to prove Weyl's Law.

Th^m 192.1 (Weyl's Law)

Let Ω be a bdd open set in $\mathbb{R}^m$. Let $N_D(\Omega,\lambda)$ be the dimension of the range of the spectral projection $P_{[0,\lambda)}$ for $-\Delta_D^\Omega$. Then if Ω is contented,

(192.2) $$\lim_{\lambda\to\infty} N_D(\Omega,\lambda)/\lambda^{m/2} = \frac{\tau_m}{(2\pi)^m} W(\Omega)$$

Proof: We will show that for any n

(192.3) $$\limsup_{\lambda\to\infty} N_D(\Omega,\lambda)/\lambda^{m/2} \le (2\pi)^{-m}\tau_m W_n^+(\Omega).$$

and

(192.4) $$\liminf_{\lambda\to\infty} N_D(\Omega,\lambda)/\lambda^{m/2} \ge (2\pi)^{-m}\tau_m W_n^-(\Omega)$$

from which (192.2) follows.

Let $\Omega_n^\pm$ denote the union of cubes whose volume enters in the defn of $W_n^\pm(\Omega)$ resp., and let $\{C_{n,\alpha}^\pm\}$ be the interiors of the actual cubes themselves, so that

(192.5) $$\overline{\Omega_n^\pm} = \bigcup_\alpha \overline{C_{n,\alpha}^\pm}$$

Now by (a) and (c) of Prop 188.1

$$-\Delta_D^{\Omega} \le -\Delta_D^{\mathrm{int}(\Omega_n^-)} \le -\Delta_D^{\cup_\alpha C_{n,\alpha}^-} = \bigoplus_\alpha -\Delta_D^{C_{n,\alpha}^-}$$

where we have used Prop 184.1 in the last step. Thus by Corollary 185.1 and Lemma 187.1,

$$N_D(\Omega,\lambda) \ge \sum_\alpha N_D(C_{n,\alpha}^-,\lambda)$$

$$= (\#\alpha)\, N_D(2^{-n-1},\lambda)$$

$$= W_n^-(\Omega)\, 2^{nm}\, N_D(2^{-n-1},\lambda).$$

By Prop 179.2

(143.1) $$\lim_{\lambda\to\infty} \frac{N_D(2^{-n-1},\lambda)}{\lambda^{m/2}} = \tau_m (2\pi)^{-m}\, 2^{-nm}$$

This proves (142.4). Similarly by (a)-(c) of Prop. 188.1

$$-\Delta_D^{\Omega} \ge -\Delta_D^{\mathrm{int}(\Omega_n^+)} \ge -\Delta_N^{\mathrm{int}(\Omega_n^+)} \ge -\Delta_N^{\cup C_{n,\alpha}^+} = \bigoplus -\Delta_N^{C_{n,\alpha}^+}$$

and (142.3) follows by mimicking the above steps. $\square$.

Remark (143.2) As $-\Delta_D^{\Omega} \ge -\Delta_N^{\Omega}$, we have $N_N(\Omega,\lambda) \ge N_D(\Omega,\lambda)$ and so $\liminf_\lambda N_N(\Omega,\lambda)/\lambda^{m/2} \ge \tau_m (2\pi)^{-m} W_\lambda^+(\Omega)$

To obtain the reverse inequality we try from (189.2)

$$-\Delta_N^{\Omega} \geq -\Delta_N^{\cup C^-_{n,\alpha}} \oplus -\Delta_N^{\Omega \setminus \cup C^-_{n,\alpha}}$$

$$= \left(\bigoplus_\alpha -\Delta_N^{C^-_{n,\alpha}} \right) \oplus -\Delta_N^{\Omega \setminus \bigcup_\alpha C^-_{n,\alpha}}$$

Thus

$$N_N(\Omega,\lambda) \leq \left(\sum_\alpha N_N(C^-_{n,\alpha},\lambda) \right) + N_N(\Omega \setminus \bigcup_\alpha C^-_{n,\alpha}, \lambda)$$

$$= \#\alpha \; N_N(2^{-n-1},\lambda) + N_N(\Omega \setminus \bigcup_\alpha C^-_{n\alpha}, \lambda)$$

$$= W_n^-(\Omega)\, 2^{nm}\, N_N(2^{-n-1},\lambda) + N_N(\Omega \setminus \bigcup_\alpha C^-_{n,\alpha}, \lambda)$$

~~and we ~~obtain~~ using (180.2) the lower bound~~

$$\cancel{\lim_{\lambda \to \infty} N_N(\Omega,\lambda)/\lambda^{m/2} \geq W_n^-(\Omega)(2\pi)^{-m}\omega_m}$$

~~Thus (192.2) also holds for $N_N(\Omega,\lambda)$.~~

But we have no control on the term

$N_N(\Omega \setminus \bigcup_\alpha C^-_{n,\alpha}, \lambda)$ and the method fails!

We only have, in general, the lower bound on $\liminf_\lambda \frac{N_N(\Omega,\lambda)}{\lambda^{m/2}}$ as above. In fact it is possible to construct regions $\Omega \subset \mathbb{R}^n$ for which

$N_N(\Omega, \lambda) = \infty$ for all $\lambda > 0$! (see eg. V. Jakšić, S. Molchanov & B. Simon, Eigenvalue asymptotics of the Neumann Laplacian of Regions & Manifolds with Cusps, J. Func. Anal. 106 (1992), 59–79: they quote in particular the following result of Hempel, Seco & Simon:

Let S be a closed subset of the positive real axis. Then $\exists$ a bounded domain Ω for which

$$\sigma_{ess}(H_N^\Omega) = S.$$

If $\partial\Omega$ is sufficiently smooth, however, then (192.2) is also true for $N_N(\Omega, \lambda)$