Lecture 12

Determinants of the form $\det(1+T)$ occur all over modern mathematical physics. For example, if we consider the so-called Gaussian Unitary Ensemble (GUE) of random $N \times N$ Hermitian matrices $M = (M_{ij}) = M^*$, with probability distribution density

$$P_N(M)\,dM = \frac{1}{Z_N} e^{-\operatorname{tr} M^2}\,dM$$

where $dM$ is Lebesgue measure on the algebraically independent entries of $M$

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1\le i<j\le N} d\operatorname{Re} M_{ij} \prod_{1\le i<j\le N} d\operatorname{Im} M_{ij}$$

then under an appropriate scaling (see e.g. Mehta's book on "Random Matrices") as $N \to \infty$

$$\operatorname{Prob}\{M : M \text{ has no eigenvalues in an interval } [a,b]\}$$

(149.1) $$= \det(1-S)_{L^2(a,b)}$$

where $S$ is the so-called "sine-kernel operator"

$$S(x,y) = \frac{\sin(x-y)}{\pi(x-y)}, \qquad x, y \in [a,b]$$

$$Sf(x) = \int_a^b S(x,y)\, f(y)\, dy.$$

$S$ is, as we will see, a trace-class operator in $L^2(a,b)$ and so (149.1) is well-defined.

The following spectral theoretic result is basic.

Th$^m$ 151.1 (the Hilbert-Schmidt Theorem)

Let $A$ be a self-adjoint compact operator on a Hilbert space $\mathcal{H}$. Then $A$ has a complete orthonormal basis of eigenvectors $\{\varphi_\alpha\}$, $A\varphi_\alpha = \lambda_\alpha \varphi_\alpha$.

(Following Reed-Simon I)

Proof: For each eigenspace $\mathcal{H}_{\lambda_\alpha} = \{\varphi : (A - \lambda_\alpha)\varphi = 0\}$, choose an orthonormal basis (if $\lambda_\alpha \neq 0$, we know from Riesz-Schauder Theory, that $\dim \mathcal{H}_{\lambda_\alpha} < \infty$ and we can construct the orthonormal basis for $\mathcal{H}_{\lambda_\alpha}$ by a Gram-Schmidt orthogonalization procedure; for $\lambda_\alpha = 0$, $\mathcal{H}_{\lambda_\alpha = 0}$ has an orthonormal basis by a general Zorn's lemma argument). The collection $\mathcal{B}$ of all such eigenvectors is again orthonormal, as eigenvectors corresponding to distinct eigenvalues are orthogonal. Let $\mathcal{M}$ be the closed linear span of $\mathcal{B}$. Since $A$ is self-adjoint,

$A: \mathcal{M} \to \mathcal{M}$ and $\mathcal{M}^\perp \to \mathcal{M}^\perp$. Let $\tilde{A}$ be the restriction of $A$ to $\mathcal{M}^\perp$. Then $\tilde{A}$ is self-adjoint and compact as $A$ is. By the Riesz-Schauder Theorem, if $\lambda \neq 0$ is in $\sigma(\tilde{A})$, it is an eigenvalue of $\tilde{A}$ and thus of $A$. Therefore the spectral radius $r(\tilde{A})$ of $\tilde{A}$ is zero since the eigenvectors of $A$ are in $\mathcal{M}$. Because $\tilde{A}$ is self-adjoint, $\|\tilde{A}\| = r(\tilde{A}) = 0$ and so $\tilde{A} = 0$. Thus $\mathcal{M}^\perp = \{0\}$ since if $\varphi \in \mathcal{M}^\perp$, $\tilde{A}\varphi = 0$ which implies $\varphi \in \mathcal{M}$. Thus $\mathcal{H} = \mathcal{M}$. $\square$

<u>Remark 152.1</u> As $A = A^*$, any eigenvalue $\lambda_\alpha$ of $A$ is necessarily real.

<u>Remark 152.2</u>

If $\mathcal{H}$ is not separable, then $\ker A$ is necessarily infinite dimensional, and, in fact, not separable.

It follows from Th$^m$ 151.1, that compact operators $A$, which are not necessarily s. adjoint, have a canonical representation.

Th$^m$ 153.1 (canonical form for compact operators)

Let $A$ be a compact operator on a separable Hilbert space $\mathcal{H}$. Then there exist (not necessarily complete) orthonormal sets $\{\psi_n\}_{n=1}^N$ and $\{\phi_n\}_{n=1}^N$, $1 \leq N \leq \infty$, and positive numbers $\{\lambda_n\}_{n=1}^N$ with $\lambda_n \to 0$, so that

(153.1) $$A = \sum_{n=1}^{N} \lambda_n (\psi_n, \cdot)\, \phi_n$$

The sum in (153.1), which may be finite or infinite, converges in norm. The numbers $\{\lambda_n\}$ are called the singular values of $A$.

Remark The singular values $(\lambda_n)$ are the square roots of the eigenvalues of $A^*A$; we have $A^*A\,\psi_n = \lambda_n^2\,\psi_n$.

Note that, by the $AB \leftrightarrow BA$ relation discussed earlier (pp 88 et seq.) The squares of the non-zero singular values $\lambda_n^2$ are also eigenvalues of $AA^*$, $AA^* \phi_n = \lambda_n^2 \phi_n$.

Proof of Th^m 153.1 Since $A$ is compact, so is $A^*A$. Thus $A^*A$ is compact and s. adjoint. By the Hilbert-Schmidt theorem 151.1, there is an orthonormal set $\{\psi_n\}_{n=1}^{N}$ so that $A^*A\psi_n = \mu_n \psi_n$ with $\mu_n \neq 0$ and so that $A^*A$ is the zero operator on the subspace perpendicular to $\{\psi_n\}_{n=1}^{N}$. Since $A^*A$ is positive, i.e. $(f, A^*A f) \geq 0 \quad \forall f \in \mathcal{H}$, each $\mu_n > 0$. Let $\lambda_n = \sqrt{\mu_n} > 0$ and set $\phi_n = (A\psi_n)/\lambda_n$. A short calculation shows that the $\phi_n$'s are orthonormal and that

$$A\psi = \sum_{n=1}^{N} \lambda_n (\psi_n, \psi) \phi_n .$$

Indeed, for any $\psi \in \mathcal{H}$, as $A^*A$ has a complete ortho. set of eigenvectors

$$\psi = \sum_{\lambda_n \neq 0} (\psi_n, \psi)\psi_n + \psi_\perp$$

where $\psi_\perp \in \mathrm{Nul}(A^*A)$.

Thus

$$A\psi = \sum_{\lambda_n > 0} (\psi_n, \psi) A\psi_n + A\psi_\perp$$

$$= \sum_{\lambda_n > 0} \lambda_n (\psi_n, \psi) \phi_n + A\psi_\perp$$

But $\|A\psi_\perp\|^2 = (\psi_\perp, A^* A \psi_\perp) = 0$. Thus

$$A\psi = \sum_{\lambda_n > 0} \lambda_n (\lambda_n, \psi) \phi_n$$

Also for $1 \le j < k$, $\left\| \sum_{n=j}^{k} \lambda_n (\psi_n, \psi) \phi_n \right\|^2 = \sum_{n=j}^{k} \lambda_n^2 |(\psi_n, \psi)|^2$

$$\le \sup_{n \ge j} \lambda_n^2 \sum_{n=j}^{k} |(\psi_n, \psi)|^2 \le \sup_{n \ge j} \lambda_n^2 \|\psi\|^2$$

and so the convergence in (153.1) is in norm.
Here we have used the fact that the $\psi_n$'s are orthonormal and also for $\lambda_n, \lambda_m \neq 0$

$$(\phi_n, \phi_m) = (A\psi_n, A\psi_m) / \lambda_n \lambda_m$$

$$= (\psi_n, A^* A \psi_m) / \lambda_n \lambda_m$$

$$= \delta_{n,m} \lambda_m^2 / \lambda_n \lambda_m = \delta_{nm},$$

and so the $\phi_n$'s are orthonormal. $\square$.

We need some basic results :

We will use the following result. Recall that we say an operator $A \in \mathcal{L}(\mathcal{H})$ is positive i.e. $A \geq 0$ iff $(f, Af) \geq 0 \ \forall f \in \mathcal{H}$. In particular, $A \geq 0 \Rightarrow A$ is s. adj. (why?)

Th$^m$ 156.1 (square root lemma)

Let $A \in \mathcal{L}(\mathcal{H})$ and $A \geq 0$. Then $\exists$ a unique $B \in \mathcal{L}(\mathcal{H})$ with $B \geq 0$ and $B^2 = A$. Furthermore $B$ commutes with every bdd operator which commutes with $A$.

Proof: See Reed Simon Vol. 1, p196.

Definition 156.2 Let $A \in \mathcal{L}(\mathcal{H})$. Then $|A| = \sqrt{A^*A}$

$|A|$ is called the modulus of $A$.

Remark 156.3

If $A$ is compact and $A \geq 0$, then clearly $|A| = A$. However if $A$ is compact and not necessarily positive, then we have
$$A^*A = \sum \lambda_n^2 (\psi_n, \cdot)\, \psi_n$$

i.e.

$$|A|^2 = \sum \lambda_n^2 (\psi_n, \cdot)\, \psi_n$$

and so clearly

$$|A| = \sum_{\lambda_n > 0} \lambda_n (\psi_n, \cdot)\, \psi_n .$$

**Definition 157.1**

An operator $U \in \mathcal{L}(\mathcal{H})$ is called an isometry if $\|Ux\| = \|x\| \ \forall x \in \mathcal{H}$. $U$ is called a partial isometry if $U$ is an isometry when restricted to the closed subspace $(\ker U)^\perp$.

Thus if $U$ is a partial isometry, $\mathcal{H}$ can be written as

$$\mathcal{H} = \ker U \oplus (\ker U)^\perp$$

and also, as $\operatorname{ran} U$ is necessarily closed,

$$\mathcal{H} = \operatorname{ran} U \oplus (\operatorname{ran} U)^\perp$$

and $U$ is a unitary operator between $(\ker U)^\perp$ and $\operatorname{ran} U$.

Exercise 157.2 Show that $U^*$ is a partial isometry from $\operatorname{Ran} U$ to $(\ker U)^\perp$ which acts as the inverse of the map $U : (\ker U)^\perp \to \operatorname{ran} U$.

We now ~~prove~~ have the analog of the decomposition $z = |z| e^{i \arg z}$ for $z \in \mathbb{C}$.

**Th$^m$ 157.3**

Let $A \in \mathcal{L}(\mathcal{H})$. Then there is a partial isometry $U$ such that $A = U|A|$. $U$ is uniquely determined by the condition that $\ker U = \ker A$. Moreover, $\operatorname{ran} U = \overline{\operatorname{ran} A}$.

<u>Proof</u>: Exercise (see Reed Simon Vol. 1).

<u>Exercise 158.1</u>

If $A \in \mathcal{L}(\mathcal{H})$ is compact, identify $U$ in the polar decomposition $A = U|A|$.

We now turn to the consideration of trace class operators. Refs: Reed Simon Vol's 1 and 4. Also B. Simon, Trace Ideals and Their Applications, Cambridge Univ. Press.

<u>Th^m 158.2</u> Let $\mathcal{H}$ be a separable Hilbert space and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal basis. Then for any positive operator $A \in \mathcal{L}(\mathcal{H})$ we define

(158.3) $$\operatorname{tr} A \equiv \sum_{n=1}^{\infty} (\varphi_n, A\varphi_n)$$

The number $\operatorname{tr} A$ is called the <u>trace of $A$</u> and is <u>independent</u> of orthonormal basis chosen. The trace has the following properties:

(a) $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$

(b) $\operatorname{tr} \lambda A = \lambda \operatorname{tr} A, \quad \lambda \geq 0$

(c) $\operatorname{tr} UAU^{-1} = \operatorname{tr} A$ for any unitary $U$ 15a

(d) $0 \leq A \leq B \Rightarrow \operatorname{tr} A \leq \operatorname{tr} B$.

Proof: To show the independence of basis, let $\{g_m\}$ be another orthonormal basis and let $B$ be the (unique) positive square root of $A$, $A = B^2$.

Then

$$\sum_n (\varphi_n, A\varphi_n) = \sum_n \|B\varphi_n\|^2$$

$$= \sum_n \sum_m |(g_m, B\varphi_n)|^2$$

$$= \sum_n \sum_m |(Bg_m, \varphi_n)|^2$$

$$= \sum_m \sum_n |(\varphi_n, Bg_m)|^2$$

$$= \sum_m \|Bg_m\|^2$$

$$= \sum_m (g_m, Ag_m).$$

(a) (b) and (d) are obvious. To prove (c), note that if $\{\varphi_n\}$ is an orthonormal basis, then so is $\{U\varphi_n\}$. Thus

$$\operatorname{tr} UAU^{-1} = \sum (U\varphi_n, UAU^{-1}U\varphi_n)$$

$$= \sum (\varphi_n, A\varphi_n)$$

$$= \operatorname{tr} A. \quad \square$$

<u>Definition 160.1</u>

An operator $A \in \mathcal{L}(\mathcal{H})$ is called <u>trace class</u> if and only if $\operatorname{tr}|A| < \infty$, where $|A| = \sqrt{A^*A}$.

The family of all trace class operators is denoted $\mathcal{B}_1 = \mathcal{B}_1(\mathcal{H})$.

The basic properties of $\mathcal{B}_1(\mathcal{H})$ are given in the following.

<u>Theorem 160.1</u> $\mathcal{B}_1$ is a $*$-ideal in $\mathcal{L}(\mathcal{H})$, that is,

(a) $\mathcal{B}_1$ is a vector space

(b) If $A \in \mathcal{B}_1$ and $B \in \mathcal{L}(\mathcal{H})$ then $AB \in \mathcal{B}_1$ and $BA \in \mathcal{B}_1$.

(c) If $A \in \mathcal{B}_1$, then $A^* \in \mathcal{B}_1$

<u>Proof</u>: (a) Since $|\lambda A| = |\lambda|\,|A|$ for $\lambda \in \mathbb{C}$, $\mathcal{B}_1$ is closed under scalar multiplication. Now suppose $A$ & $B$ are in $\mathcal{B}_1$: we wish to prove that $A + B \in \mathcal{B}_1$.

Let $U, V, W$ be the partial isometries arising from the polar decompositions

$$A+B = U\,|A+B|$$
$$A = V\,|A|$$
$$B = W\,|B|$$

Then for any ortho. basis $\varphi_n$

$$\sum_{n=1}^{N} (\varphi_n, |A+B|\,\varphi_n) = \sum_{n=1}^{N} (\varphi_n, U^*(A+B)\,\varphi_n)$$

(as $U^*U|A+B| = |A+B|$ : why?)

$$= \sum_{n=1}^{N} (\varphi_n, U^*V|A|\varphi_n) + U^*W|B|\varphi_n)$$

$$\leq \sum_{n=1}^{N} |(\varphi_n, U^*V|A|\varphi_n)|$$

$$+ \sum_{n=1}^{N} |(\varphi_n, U^*W|B|\varphi_n)|$$

However,

$$\sum_{n=1}^{N} |(\varphi_n, U^*V|A|\varphi_n)| \leq \sum_{n=1}^{N} \||A|^{\frac{1}{2}}V^*U\varphi_n\| \, \||A|^{\frac{1}{2}}\varphi_n\|$$

$$\leq \left(\sum_{n=1}^{N} \||A|^{\frac{1}{2}}V^*U\varphi_n\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \||A|^{\frac{1}{2}}\varphi_n\|^2\right)^{\frac{1}{2}}.$$

Thus if we can show

(161.1) $$\sum_{n=1}^{N} \||A|^{\frac{1}{2}} V^*U\varphi_n\|^2 \leq \operatorname{tr}|A|$$

we can conclude that

$$\sum_{n=1}^{N} (\varphi_n, |A+B|\varphi_n) \le \operatorname{tr}|A| + \operatorname{tr}|B| < \infty$$

and thus $A+B \in \mathcal{B}_1$. But to prove (161.1), we need only to prove that

$$\operatorname{tr} U^* V |A| V^* U \le \operatorname{tr}|A|.$$

Picking an orthonormal basis $\{\varphi_n\}$ with each $\varphi_n$ in $\ker U$ or in $(\ker U)^\perp$ we see that

$$\operatorname{tr} U^* V |A| V^* U = \sum (\varphi_n, U^* V |A| V^* U \varphi_n)$$

$$= \sum (U\varphi_n, V|A|V^* U\varphi_n)$$

$$\le \operatorname{tr}(V|A|V^*)$$

Similarly picking an orthonormal basis $\{\psi_m\}$ with each $\psi_m$ in $\ker V^*$ or $(\ker V^*)^\perp$ we find $\operatorname{tr} V|A|V^*$ $\le \operatorname{tr}|A|$.

Lecture 13 (b) By the lemma proven below, each $B \in \mathcal{L}(\mathcal{H})$ can be written as a linear combination of 4 unitary operators and