Intro. to Math. Modeling PROBLEM SET 8 Due March 30, 2005

1. The traffic flow can be modeled with $q = 70\rho(1 - \rho/377)$ vehicles per hour. The traffic is moving at a constant speed and a constant density of 250 vehicles/mile. Because of an overturned truck, cars start to slow down slightly, causing a slight density increase. At $t = 0$ we thus observe that the density is 250 when $x < 0$ and is 270 when $x > 0$.

(a) How fast are cars moving before they encounter the wave?

(b) Using linear theory with $\rho_0 = 250$, estimate the velocity of the traffic wave. For what t (in minutes) will the car which was located at $x = -1$ mile at $t = 0$ encounter the wave? Sketch the wave propagation in an $x - t$ diagram.

(c) Show from the general theory that a driver in the pack at density 250 will see the traffic wave approaching at speed $u_{max}\rho/\rho_{max}$. Check this against the numbers you gave in (a) and (b).

2. Solve the following linear first-order PDEs with the indicated initial condition. In each case verify that you have a solution by substitution back in the equation.

$$(a)\ \frac{\partial f}{\partial t} + \frac{xt}{1+t^2}\frac{\partial f}{\partial x} = 0, \quad f(x,0) = \sin(x),$$

$$(b)\ \frac{\partial f}{\partial t} + \frac{1}{1+x}\frac{\partial f}{\partial x} = 0, \quad f(x,0) = x.$$

In (b) assume $x > -1$. (Hint in (b): $F(\phi) = -1 + \sqrt{1+2\phi}$.)

3. Problem 71.1, page 322 of text. (a is a positive constant, and t is measured in hours.)

4. Problem 71.2, page 322 of text.

5. Apply the method we have used to solve the nonlinear traffic flow equation to the equation

$$\frac{\partial \rho}{\partial t} + \rho^2\frac{\partial \rho}{\partial x} = 0, \quad x > 0,$$

with the initial condition $\rho(x,0) = x > 0$. Verify your answer by substituting in the equation. (Hint: $x = \rho^2(x_0,0)t + x_0$.)

6. For the red light problem with $q = u_{max}\rho(1 - \rho/\rho_{max})$, show that the expansion fan in the transition region has the form

$$\rho(x,t) = \rho_{max}\left(\frac{u_{max}t - x}{2u_{max}t}\right).$$