Intro. to Math. Modeling PROBLEM SET 5 Due Wednesday, February 23, 2005

1. Problem 49.1, page 227 of text. Explain the interactions by drawing a "depress-enhance" diagram.

2. Consider the host-parasite interaction in which the host growth rate is density dependent in the logistic sense. The equations are then

$$\frac{dP}{dt} = -dP(1 - CH), \quad \frac{dH}{dt} = bH(1 - AH - BP),$$

where A, B, C are positive constants with $A < C$.

(a) What are the possible equilibria for this system? (There are three distinct equilibrium pairs (P_e, H_e).)

(b) With $A = 1, B = C = 2$, linearize the resulting system about the equilibrium having positive population for both host and parasite. Give the pair of linear first-order equations determining the stability of this equilibrium.

3. Consider the pair of population equations

$$\frac{dP}{dt} = \frac{-P}{1 + H}, \quad \frac{dH}{dt} = \frac{H}{1 + P}.$$

(a) If $P(t), H(t)$ solve this system, show that

$$J(t) = \frac{P(t)H(t)}{(1 + P(t))(1 + H(t))}$$

satisfies $\frac{dJ}{dt} = 0$, i.e. J is constant on trajectories of the system in the phase plane. (Hint: Compute $\frac{dJ}{dt}$ using the chain rule, i.e. $\frac{dJ}{dt} = \frac{\partial J}{\partial P}\frac{dP}{dt} + \frac{\partial J}{\partial H}\frac{dH}{dt}$, and use the two equations.)

(b) Derive this property of J by integrating the phase plane equation

$$\frac{dH}{dP} = -\frac{H(1 + H)}{P(1 + P)}.$$

(Hint: Use separation of variables.)

4. (This is basically problem 49.2, page 227.) A two-species ecological system is described by

$$x_{m+1} - x_m = \Delta t x_m(2 - y_m), \quad y_{m+1} - y_m = \Delta t y_m(-3 + x_m).$$

Here x_m, y_m are the populations at time $m\Delta t$.

(a) Briefly describe how you might interpret the interaction of the species.

(b) Find all equilibrium populations.

(c) Show that the perturbations of the equilibrium $(x_e, y_e) = (3, 2)$ satisfy

$$\delta x_{m+1} - \delta x_m = -3\Delta t \delta y_m, \quad \delta y_{m+1} - \delta y_m = 2\Delta t x_m.$$

(d) If x_m is eliminated from this system, show that $y_{m+2} - 2y_{m+1} + \alpha y_m = 0,\ \alpha = 1 + 6\Delta t^2$. (Hint: Consider the equation for y_m for both m and $m + 1$.)

(e) Solve this difference equation and verify that the populations will grow and oscillate. (Hint: Consider a trial solution $y_m = \lambda^m$ and show that λ is complex and that $|\lambda| > 1$.)