Intro. to Math. Modeling **PROBLEM SET 11** **Due April 20, 2005**

Note: This is the last graded problem set. A set of review problems will be handed out next week.

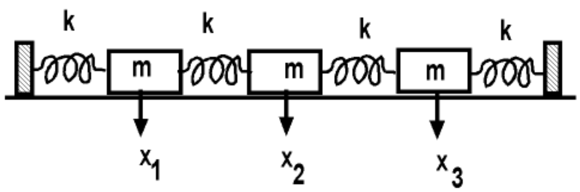
1. (a) A mass of ten grams is attached to a spring and allowed to hang without moving in the Earth's gravitational field (g=980 cm/sec^2). When it is attached , the spring is found to stretch 1 cm. Assuming that the spring satisfies Hooke's law, and that friction is absent. At what frequency (in Hertz) will the mass oscillate when it is set in motion?

(b) Suppose the device is carried to the moon, where gravity is 1/6 that of earth. How much is the spring stretched on the moon (no oscillation)? What is the frequency of oscillation on the moon?

2. In the $x, m\dot{x}$ phase plane (or $(q,p)-$ plane) of the simple harmonic oscillator, the curves of constant energy $E = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2$ are ellipses. The area of an ellipse with semi-axes a, b is πab. Prove that the period T of the oscillator is given by $\frac{dA}{dE}$ where A is the area enclosed by the ellipse for this energy.

Bonus question (This is optional, but counts as a full problem). Prove that for any nonlinear oscillator with potential V such that $m\frac{d^2x}{dt^2} + dV/dx = 0$, the same relation applies: $T = dA/dE$ where A is the area within the closed integral curve carrying energy E.

3. Consider three equal masses m connected by springs satisfying Hooke's law and having the same spring constant k, as shown in the figure. (There is no friction.)



(a) We need to write down the system of three equations for x_1, x_2, x_3, the three displacements from equilibrium. The first of these is

$$m\frac{d^2x}{dt^2} = -kx_1 + k(x_2 - x_1).$$

What are the other two equations?

(b)Clearly one mode of oscillation is where the center mass stays stationary. Show that for this mode the frequency of oscillation is $sqrt 2k/m$.

(c) Write the three equations as a vector equation $\frac{d^2X}{dt^2} + \frac{k}{m}AX = 0$.

(d) Determine the three frequencies of oscillation from the roots of $Det(A - \lambda^2 I)$. Remember that we already know one root from (a)($\lambda^2 = -2k/m$).

(e) Determine the eigenvectors of corresponding to the three frequencies, and describe what the masses are doing in each of the two new oscillations.

(OVER)

4. Consider the nonlinear equation

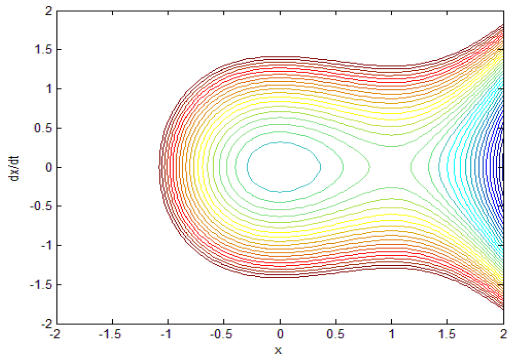
$$\frac{d^2x}{dt^2} + x - 2x^2 = 0.$$

(a) What is $V(x)$ for this problem?

(b)The phase plane, i.e. the curves of constant E for this solution, is shown in the figure below. Redo the sketch indicating with arrows the direction of motion on the integral curves as t increases.

(c) Using E-V analysis, determine a value of E above which the device will not oscillate continuously. Give an example of initial conditions, $x(0), \dot{x}(0)$, such that the system will oscillate, and give the energy E of this oscillation.

(d) What happens if $x(0) = 0, \dot{x}(0) = -1$?



5. To make a good clock, the device should oscillate with a fixed period, and if disturbed should return after a transient motion to the same oscillation with the same period. Consider a device which in the phase plane $(x(t), y(t) = m\dot{x})$ moves according to the the following equations *in polar coordinates* r, θ, where $r^2 = x^2 + y^2, \theta = arctan(y/x)$:

$$\frac{dr}{dt} = r(1-r), \ \frac{d\theta}{dt} = 2\pi$$

In the following you can if you want use qualitative reasoning with a careful sketch, rather than solve a logistic equation:

(a) What happens if the system starts from $\theta = 0, r = .5$?

(b) What happens if the system starts from $\theta = 0, r = 1.5$?

(c) Describe the behavior of the system starting from any initial condition as $t \to \infty$.

(d) Would the system make a good clock?

(e) If time is in seconds, how many cycles would the clock go through in one minute?