Notes on mechanical vibrations (continued)

Stephen Childress

April 17, 2005

5 The pendulum

An ideal pendulum is a straight rigid, massless bar attached to a frictionless hinge at one end and to a mass m at the other end.The mass or bob may be taken as a sphere with center a distance L from the hinge. The hinge determines a plane in which motion takes place. The pendulum moves under the action of a uniform gravitational field acting downward. Thus the mass will move on a circle of radius L and center at the hinge, see figure 7.

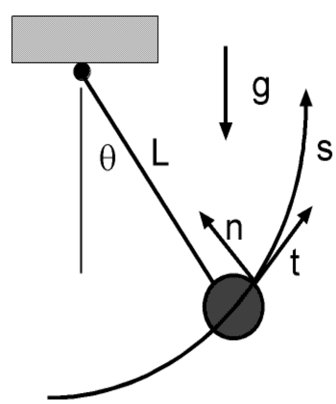

Figure 7. A simple pendulum

As the mass moves on the circular path, its direction as well as its speed will change, i.e. its velocity is a function of time, so acceleration is involved. By Newton's second law of motion, a force must be applied. Part of the force is applied by the bar and acts radially. The force acting tangential to the path, and therefore perpendicular to the bar, cannot be exerted by the bar and so must be entirely due to gravity. Let $\mathbf{t}$ be the unit tangent vector to the circular path of the bob, and let s denote arc length along the path. Then from the calculus of plane curves we have

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \tag{33}$$

where $\kappa =$ curvature, here the inverse of the rod length L, $\kappa = L^{-1}$, and $\mathbf{n}$ is the unit vector normal to the path, see figure 7.

Another equation of plane curves is

$$\frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} = -L^{-1}\mathbf{t}. \tag{34}$$

We will use these relations to compute acceleration of a point moving along the path with position $s(t)$. Let $ds/dt = q$ be the speed of motion along the path. Then the velocity is $\mathbf{v} = q\mathbf{t}$, and the acceleration is

$$\frac{d\mathbf{v}}{dt} = \frac{dq\mathbf{t}}{dt} = \frac{dq}{dt}\mathbf{t} + q\frac{d\mathbf{t}}{ds}\frac{ds}{dt} = \frac{dq}{dt}\mathbf{t} - q^2L^{-1}\mathbf{n}. \tag{35}$$

Now the term on the right involving the acceleration in the direction $\mathbf{n}$, must be caused by a force exerted by the bar. The part involving $\mathbf{t}$ must be associated with the force of gravity. Now if s is zero when the pendulum is hanging straight down, then $s = L\theta$ where θ is the angle in figure 7 expressed in radians. Thus $q = L\frac{d\theta}{dt}$. Since the force of gravity has a component $-mg\sin\theta$ in the direction of $\mathbf{t}$, Newton's law for the tangential motion is

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta, \tag{36}$$

or

$$m\frac{d^2\theta}{dt^2} + k\sin\theta,\ k = \frac{mg}{L}. \tag{37}$$

Note that this is an equation in the general form of our nonlinear oscillator equation with x replaced by θ and V by $k\sin\theta$. We then see that the energy equation takes the form

$$\frac{m}{2}\Big(\frac{d\theta}{dt}\Big)^2 + k(1-\cos\theta) = E, \tag{38}$$

where we have included a constant to make the potential energy equal to zero when the bob is at its lowest point.

We show the phase plane for the pendulum, based on (38), in figure 8. The general interpretation of the integral curves follows our previous examples. The points A, B mark the extremes $\theta = \pm\pi$, where the pendulum has an unstable equilibrium. The curve at C represents a steady increase of θ to the right. This corresponds to the pendulum moving continually around the hinge (we must assume that the hinge allows full movement around the circle) in a clockwise direction. We can see that this occurs for energies greater than $2k$. Indeed, from (38) if $E > 2k$ we have $E - k(1-\cos\theta) > 0$, so $d\theta/dt$ cannot vanish. Similarly if $E < 0$ the pendulum revolves steadily in the counter-clockwise direction.

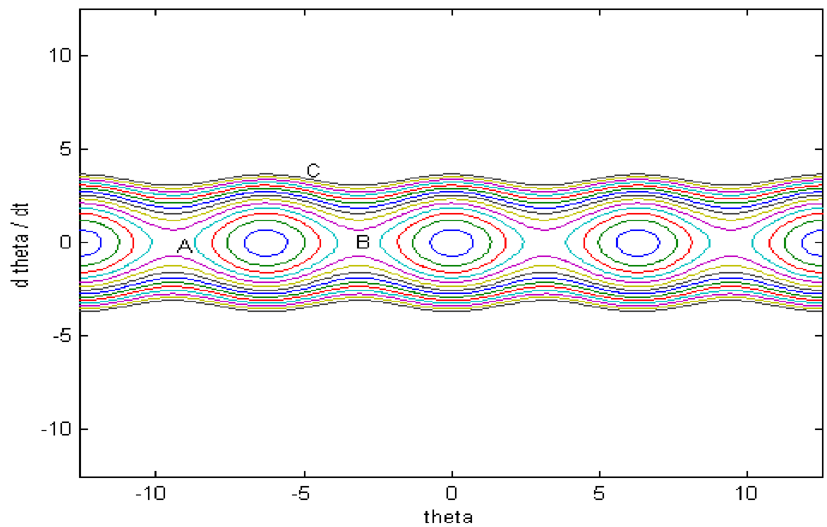

Figure 8. The phase plane of the simple pendulum: lines of constant E given by (38) for $m = k = 1$.

This behavior can be also approached using $E - V$ analysis of a bead or ball sliding without friction on a sinusoidal curve under the action of gravity. Given a sufficient push it keeps going over hill after hill.

We have written (38) in a way that allows us to see easily that the pendulum reverts to a simple harmonic oscillator when θ remains small, i.e. the pendulum oscillates near the bottom of its arc. Since

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + O(\theta^4), \quad |\theta| \ll 1, \tag{39}$$

we see that (38) becomes

$$\frac{m}{2}\Big(\frac{d\theta}{dt}\Big)^2 + \frac{k}{2}\theta^2 = E, \tag{40}$$

which agrees with the energy relation of a simple harmonic oscillator. Thus for small amplitudes the period of a pendulum doesn't change, but for finite amplitudes it does. If the oscillation is between 0 and θ_{max}, the period is

$$T = 2\sqrt{2m}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{E - k(1-\cos\theta}}. \tag{41}$$

Note that $E - k(1 - \cos\theta_{max}) = 0$. We can then rewrite this expression using $\cos^2(\theta/2) = \frac{1}{2}(1-\cos\theta)$ as

$$T\sqrt{k/m} = 4\int_0^{\phi_m} \frac{d\phi}{\sqrt{\sin^2\phi_m - \sin^2\phi}}, \tag{42}$$

where $\theta_{max} = 2\phi_m$. We show this relation in figure 9. Note that for small θ_{max} the period is about $2\pi\sqrt{m/k}$, but as the amplitude increases it gets larger, in fact tending to infinity as $\theta_{max} \to \pi$. This corresponds to the maximum amplitude approaching the equilibrium at $\theta = \pi$.

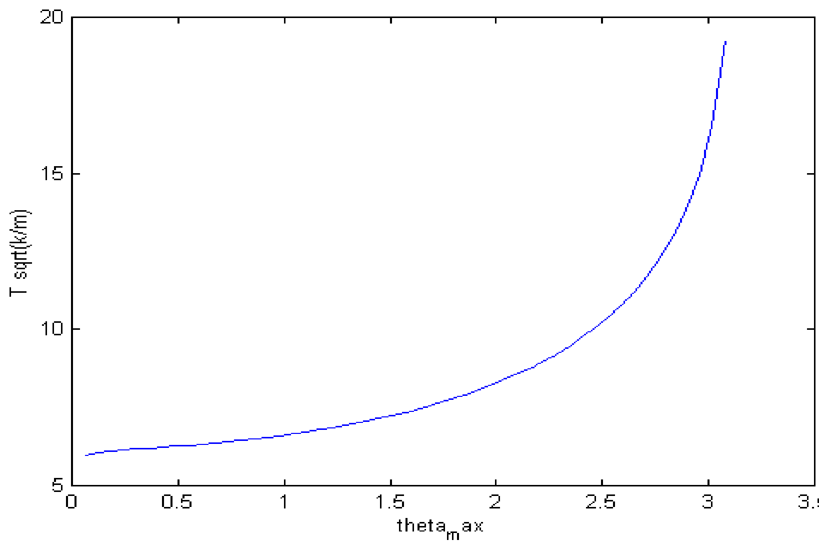

Figure 9. The period of the nonlinear pendulum as a function of θ_{max}.

5.1 A pendulum oscillator with period independent of amplitude

To discuss other physical realizations of nonlinear oscillators in a gravitational field it is helpful to change from the pendulum model to one of a "bead on a frictionless wire". A bead slides on a wire whose path is $y(x)$ and corresponding potential energy $V(x = gmy(x)$, just as in $E - V$ analysis. Note that the nonlinear pendulum can be regarded in this way as a case where the wire is a circular arc. The question is, *can the wire be deformed from a circle in such a way that the period of oscillation is independent of amplitude?*. In other words can we avoid the dependence shown in figure 9 by suitably deforming the path from a circular shape?.

It will be convenient to let the wire position be given in the form $y = y(s)$ where s is arc length measured from the lowest point. We shall assume the shape we are looking for is symmetric in x, so that the period will be four times the time to go once from the lowest point to the highest point. Let the bead move according to $s = s(t)$. Then the speed is ds/dt and, by conservation of energy,

$$\frac{m}{2}(\frac{ds}{dt})^2 + mgy(s) = E. \tag{43}$$

Thus the period is given by

$$T = 4\sqrt{m/2}\int_0^{s_m} (E - mgy(s))^{-1/2} ds, \tag{44}$$

where $mgy(s_m) = E$. We would like this integral to be independent of E. To do this, note that

$$\int_0^{\sqrt{a/b}} \frac{ds}{\sqrt{a - bs^2}} = \frac{1}{\sqrt{b}}\int_0^1 \frac{du}{\sqrt{1-u^2}} du = \frac{\pi}{2\sqrt{b}}. \tag{45}$$

Thus the integral is independent of a. Thus if we set $y = Cs^2$ in (44) (C being a positive constant) and use (45) we get

$$T = 4\sqrt{m/2}\int_0^{\sqrt{\frac{E}{mgC}}} (E - mgCs^2)^{-1/2} ds = 2\pi/\sqrt{gc}, \tag{46}$$

which is indeed independent of E and hence of the amplitude.

The equation for the wire shape is thus $y(s) = Cs^2$. Taking the square root,

$$\sqrt{y} = \sqrt{C}s = \sqrt{C}\int^x \sqrt{1 + y_x^2} dx. \tag{47}$$

Taking the derivative,

$$\frac{1}{2}\frac{1}{\sqrt{y}} y_x = \sqrt{1 + y_x^2}. \tag{48}$$

Squaring, rearranging, and solving for dy/dx we get

$$\frac{dy}{dx} = \sqrt{\frac{4Cy}{1 - 4Cy}}. \tag{49}$$

Thus

$$\int \sqrt{\frac{1 - 4Cy}{4Cy}} dy = x. \tag{50}$$

This integral may be found by the substitution $y = \frac{1}{4C}\sin^2(\theta/2)$, yielding

$$x = K(\theta + \sin\theta),\ y = K(1 - \cos\theta),\ K = \frac{1}{8C}. \tag{51}$$

This curve is a *cycloid*, and is shown in figure 10. Note that it steepens to keep the period from increasing as in the case of the pendulum.

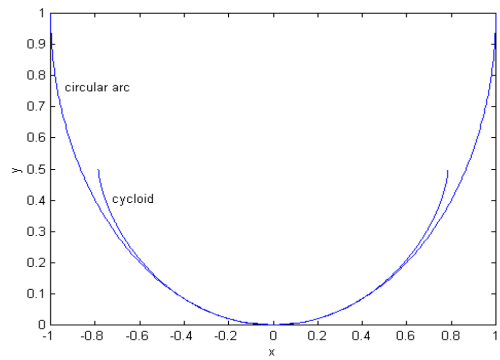

Figure 10. The path of a nonlinear pendulum compared to that of the cycloid of constant period.

6 The limit cycle

A good clock has to keep good time. The cycle it is in should be stable, in the sense that it will not only correct from any disturbance, but also that during the correction it does not lose or gain time. Real clocks have friction, and so need a source of energy to keep going. A pendulum clock has such a source in the potential energy of the weights which drive the mechanism. A pendulum clock in fact has two stable operating states. one in which the pendulum hangs motionless at the bottom rest state. The other is obtained by "starting the pendulum to swing". Once going the escapement mechanism provides a source of energy to maintain the swing. Slight perturbations from the cycle die away. Also there are often corrections to the way the pendulum is attached to make its path a cycloid rather than a circular arc, making deviations from the basic period extremely small even in the presence of small changes in amplitude.

The qualitative form of the phase plane of a pendulum clock is shown in figure 11.

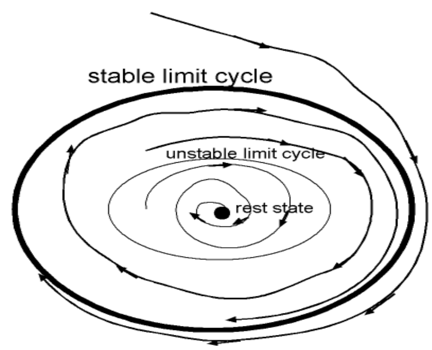

Figure 11. The phase plane of a real pendulum clock.

In problem set 11 we gave an example of a limit cycle determined by a system of two ODE's. We now give another famous example which comes from the construction of electronic oscillators. The *Van der Pol oscillator* is defined by the system

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = \mu(1 - y^2)v - y. \tag{52}$$

We show the phase plane in figure 12.

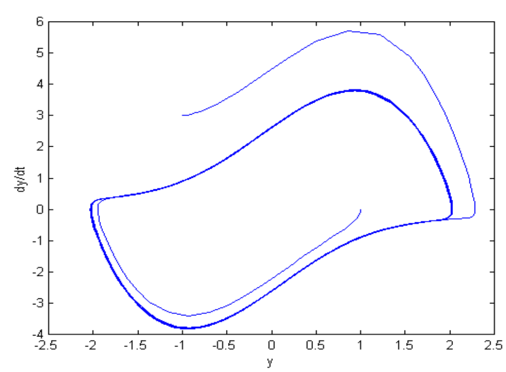

Figure 12. The phase plane of the Van der Pol oscillator with $\mu = 2$. the limit cycle is reached by two integral curves on either side of it.

6.1 The Poincaré-Bendixson theorem

One might then ask under what conditions a system of two equations possesses a limit cycle to which all integral curves tend. A remarkable fact is that this is a property which is easily obtained in a situation where there are no equilibrium points which can "attract" integral curves. The Poincaré-Bandixson theorem may be stated as follows:

Consider a system of the form

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y), \tag{53}$$

defined in an annular region between two simple closed curves C_- and C_+, where C_+ contains C_-. We suppose that within the annular region there are no equilibrium points. We further assume that on C_-, the integral curves are pointing into the annular region at time increases, and that on the outer curve C_+, the integral curves are pointing into the region. Then the theorem states that every orbit must spiral toward a simple close curve, which is itself a periodic solution of the system and represents a stable limit cycle. This situation is illustrated in figure 13.

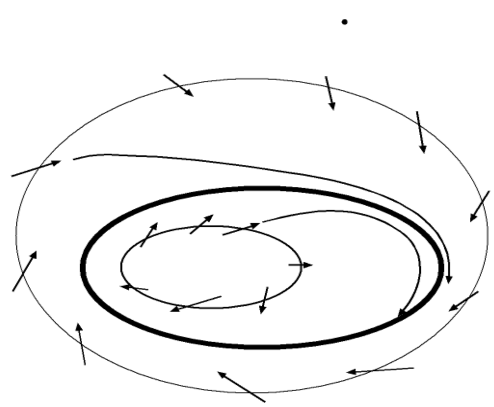

Figure 13. The Poincaré-Bendixson theorem asserts the existence of a stable limit cycle in two dimensions under the conditions illustrated here.