

Notes on mechanical vibrations

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1 Masses and springs- the linear oscillator

These notes will be related to sections 1-29 of the text. I will indicate between brackets [...] references to the sections closest to the discussion here.

We are will study mechanical vibrations for two reasons: on the math side, it will introduce second-order ODEs (involving two differentiations) in the context of Newton's equations of motion. We will revisit the idea of a phase plane in two variables, as we have used it in the study of two-species population models, but now with a physical meaning derived from Newtonian physics.

The centerpiece of Newtonian physics is the second law [1-3,5], stating that a body of mass m will accelerate when a force \mathbf{F} is applied to it, in such a way that

$$m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}. \quad (1)$$

Here $\mathbf{x}(t)$ is the position of the mass- for simplicity think of a uniform sphere with \mathbf{x} denoting the position of its center. (In general \mathbf{x} will denote the *center of mass* of the body.). A better way to state this law is in terms of *momentum*. The momentum of the mass is $m\mathbf{v}$ where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the *velocity* of the body (of it's center of mass). Then Newton's second law becomes

$$\frac{d m \mathbf{v}}{dt} = \mathbf{F}, \quad (2)$$

i.e. it states that the *rate of change of the momentum of the body is equal to the instantaneous force applied to the body.*

In addition to a mass, we also want to introduce a spring with special properties, then use such springs to set up systems of masses which oscillate. Let a mass rest on a slippery plane (no friction). Note that using "rest" implies we are thinking about an experiment on Earth in its gravitational field. This field pushes the mass against the wall and gives rise to friction, which we want to neglect here. We consider motion of a mass along a line(the x -axis), and suppose the mass is attached to a spring, which is in turn attached to a firm support. The mass will be placed so that the spring is unstretched, that is does not exert any force on the mass. We now want to make an oscillation by stretching the spring and releasing the mass. We assume that the spring has

a linear property usually referred to as *Hooke's law*: the force exerted by the spring is proportional to the distance the spring is stretched from its position of zero force (rest position). Let x denote the displacement of the center of mass of the body from its rest position. This force will pull the mass back when it is displaced to the right (positive x) see figure 1.

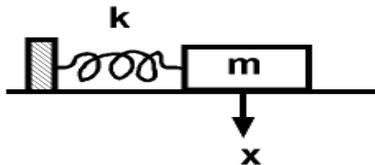


Figure 1. Frictionless spring-mass system.

We let the positive constant k (the *spring constant*) be the constant of proportionality between the displacement and the force of the spring. Thus Newton's second law for the mass is

$$m \frac{d^2 x}{dt^2} = -kx, \quad (3)$$

or

$$\frac{d^2 x}{dt^2} + \frac{k}{m} x = 0. \quad (4)$$

Since $\frac{d^2}{dt^2} \sin \omega t = -\omega^2 \sin \omega t$. We see that $\sin \omega t$ is a solution of the second-order ODE given by (4) provided that $\omega^2 = k/m$. We also can allow an arbitrary phase ϕ and amplitude A and so the general solution of (4) is

$$x = A \sin(\sqrt{k/m} t + \phi). \quad (5)$$

We can call this *simple harmonic motion*. It can also be called a *sinusoidal oscillation* or although of course for a general ϕ is a combination of sin and cos,

$$\sin(\sqrt{k/m} t + \phi) = \sin(\sqrt{k/m} t) \cos \phi + \cos(\sqrt{k/m} t) \sin \phi. \quad (6)$$

We also want to be able to use the complex notation for the general solution

$$x = A e^{i \sqrt{k/m} t}. \quad (7)$$

The quantity $\omega = \sqrt{k/m}$ has the following meaning: if t increases by $2\pi/\omega$, then $t\omega$ increases by 2π , so that $A \sin(\omega t + \phi)$ goes through one cycle. Thus $T = \frac{2\pi}{\omega}$ is the *period* of the oscillation. Conversely, if T is the period of oscillation in *seconds/cycle*, $T^{-1} = \frac{\omega}{2\pi}$ is the number of cycles per second of the oscillation, usually referred to as the unit *Hertz*, 10 Hertz being 10 oscillations/sec. In the physics literature the symbol f is most often used for frequency in hertz, so we have $\omega = 2\pi f$.

1.1 A spring-mass in gravity

Now suppose that the spring is vertical and the Earth's surface gravitational field is present [4]. If no mass is attached, it takes up its rest length. If this mass is not oscillating, it will have stretched the spring and come to a new rest state. If the gravitational constant is g , then a mass m will experience a force gm and so the spring will have been stretched by this force a distance Δx where $mg = k\Delta x$. Let's consider an example and pay attention to the units of these quantities [6]. For the earth at its surface, g is approximately 980 cm/sec^2 . This means that if I drop a mass m in free fall, it will be subjected to force gm , and then by Newton's law to a constant acceleration of $980 \text{ cm}^2/\text{sec}$. It will therefore travel $980t^2/2$ centimeters in t seconds, or 490 cm of about 16.1 feet in the first second.

Suppose the mass is 2 grams and the spring is found to be stretched .5 centimeter when it hangs. This impels

$$k = \frac{2 \cdot 980}{.5} = 3920 \text{ gram/sec}^2. \quad (8)$$

Thus $\omega = \sqrt{k/m} = \sqrt{3920/2} = 44.27 \text{ sec}^{-1}$, so the period is $T = \frac{2\pi}{\omega} = .142$ seconds. Thus there are $1/.142 = 7.05$ cycles per second when the mass oscillates on the spring, a frequency of 7.05 hertz.

1.2 Kinetic energy and work

If we multiply (3) by $\frac{dx}{dt}$ we get

$$\frac{dx}{dt} m \frac{d^2x}{dt^2} + k \frac{dx}{dt} x = \frac{d}{dt} \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 + \frac{k}{2} x^2 \right] = 0, \quad (9)$$

or

$$\frac{m}{2} \left(\frac{dx}{dt} \right)^2 + \frac{k}{2} x^2 = \text{constant} = E. \quad (10)$$

We term the constant E the *total energy* of the mass-spring system. The reason for this term is the physical meaning of the two parts of E . Perhaps the most basic principle of physics is the *conservation of energy* in all of its various forms. This is an abstract principle but each form which energy can take is very real. *Kinetic energy* is the energy possessed by a body of given mass m in motion with a given velocity \mathbf{v} , and it equals $E_{kin} = \frac{m}{2}v^2$. The *potential energy* of a spring obeying Hooke's law is equal to the work done to compress or stretch the spring. Thus

$$E_{pot} = \int_0^x kx dx = \frac{k}{2}x^2. \quad (11)$$

Thus we see that $E = E_{kin} + E_{pot} = \text{constant}$ is a statement of the conservation of energy for this system. As the mass oscillates, the energy of the system of mass in springs changes from being stored as potential energy of the spring (when velocity is zero at each end of the oscillation) to being realized as kinetic energy of the mass (as x passes through zero).

Geometrically, this exchange of energy may be graphed in the phase plane with ordinate $m\dot{x}$ as the momentum and abscissa x the position, see figure 2. By the way we will sometime write the phase plane using the traditional notation q (in place of x for the coordinate, and p in place of $m\dot{x}/dt$ for the momentum).

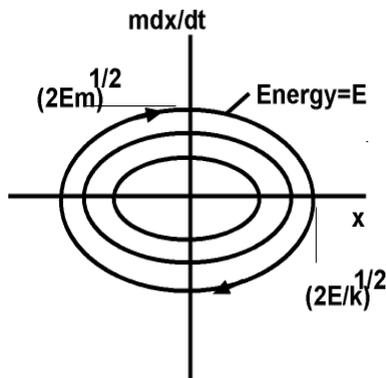


Figure 2. The phase plane of a simple harmonic oscillator.

1.3 The initial-value problem

We can ask how we might actually start an oscillator in motion [8]. We can pull and hold, that is, start with zero velocity but with the spring in a stretched state, or else the position can be the rest position and we can give the mass a kick, so initially x is zero but the velocity is not. In fact we must give both the initial velocity and the initial position in order to uniquely determine the particular solution of the problem from our general solution.

If we write the general solution in the form $x = A \sin \omega t + B \cos \omega t$ and if the initial time is $t = 0$, then clearly $B = x(0)$ and $A\omega = dx/dt(0)$. Thus the constants are uniquely determined by the two initial conditions.

2 Oscillation of two coupled masses

We now consider a two-mass oscillator. Some related problems are considered in [9]. Our purpose is to illustrate how linear algebra can help to analyze mechanical oscillations more complex than the simple harmonic oscillator. We consider the setup shown in figure 3.

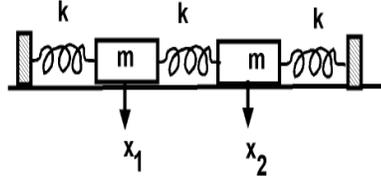


Figure 3. Oscillation of two identical masses connected by three identical springs.

There will now be two equations for the positions x_1, x_2 of the two masses. Note that these represent the deviations of the positions from the rest or equilibrium state. To get the equations satisfied by x_1, x_2 we need to account for the stretching of all springs attached to the mass. Thus we see that the following two equations result:

$$m \frac{d^2 x_1}{dt^2} = -kx_1 + k(x_2 - x_1), \quad m \frac{d^2 x_2}{dt^2} = -kx_2 - k(x_2 - x_1). \quad (12)$$

Letting

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad (13)$$

we have the vector equation

$$m \frac{d^2 X}{dt^2} = kA \cdot X = 0. \quad (14)$$

Just as we have done for systems of two linear first-order ODEs, we can find solutions to this problem by looking for the exponential solutions $X = e^{\lambda t} X_0$ where X_0 is a constant vector. Then we have a determinant to be set equal to zero,

$$\text{Det}(A - \lambda^2) = 0. \quad (15)$$

With $k/m = \omega^2$, the determinant takes the form

$$(2 + (\lambda/\omega)^2)^2 = 1 \quad (16)$$

The roots are seen to be

$$\lambda = \pm i\omega, \quad \pm i\sqrt{3}\omega. \quad (17)$$

To see what these two oscillations are doing, we can consider the eigenvectors. if $\lambda = \lambda_1 = i\omega$, the eigenvector X_1 satisfies

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} X_1 = 0, \quad (18)$$

so we may take

$$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (19)$$

Thus the two masses move together. The middle spring doesn't compress at all, so we have two masses stretching and compressing the two end springs in a symmetrical way. The frequency of oscillation is that of a single simple harmonic oscillator.

if $\lambda = \lambda_2 = i\sqrt{3}\omega$, we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X_2 = 0, \quad (20)$$

so we may take

$$X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (21)$$

Now the masses move in opposition. All three springs are involved, and the frequency of oscillation in the second mode is larger by a factor $\sqrt{3}$ than for the first mode.

3 Nonlinear oscillators

The methods we have introduced here can be applied to nonlinear systems which oscillate. The general system we would like to consider will have the form

$$m \frac{d^2x}{dt^2} + \frac{dV}{dx}(x) = 0. \quad (22)$$

Here $V(x)$ is some differentiable function, called the *potential energy*. For the simple harmonic oscillator $V = kx^2/2 = E_{pot}$, and the same physical significance attaches to V generally. Indeed if we multiply (22) by $\frac{dx}{dt}$ and integrate, we have

$$E = m \left(\frac{dx}{dt} \right)^2 + V(x) = \text{constant}. \quad (23)$$

Consider the example

$$V(x) = kx^2/2 - x^4. \quad (24)$$

The corresponding ODE is

$$m \frac{d^2x}{dt^2} + kx - 4x^3 = 0. \quad (25)$$

Because of the x^3 term this is a *nonlinear ODE of second order*. Just as for the simple harmonic oscillator, the curves of constant E , now given by

$$E = \frac{m}{2} \left(\frac{dx}{dt} \right)^2 + kx^2/2 - x^4, \quad (26)$$

are the curves along which the system moves, i.e. are the integral curves for the solutions of the equation.

We show the phase plane for (25) in figure 4, for the case $m = k = 1$.

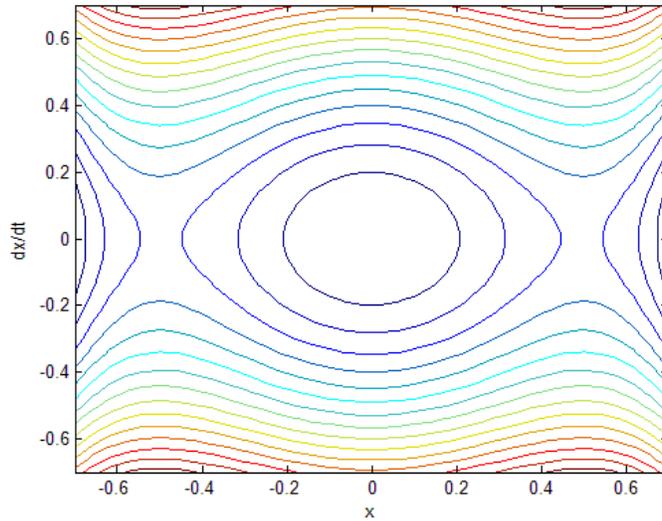


Figure 3. Phase plane of the equation (25), with $m = k = 1$.

3.1 $E - V$ analysis

We now describe how we can understand the phase plane of figure 3 using a simple analogy. In figure 4 we sketch the function $V(x)$ with $k = 2$. Since $E - V = \frac{m}{s}(dx/dt)^2$, we see that if x is changing in time then $E > V$ at that value of x . Also, in order to oscillate, there must be two zeros of $E - V$ in each cycle, corresponding to the extremes of the oscillation. For x where $E > V$, the difference $E - V$ is the kinetic energy, and hence tells us the speed at that point is $\sqrt{2m(E - v)}$. Thus an *oscillation* is determined by any local minimum of V and an energy the “potential well” thus formed in two points, see the line AB of figure 4.

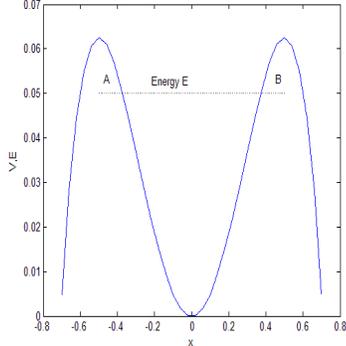


Figure 4. $E - V$ analysis of the equation (25), with $k = 1$.

A nice way to think of this is as an analogy to a marble rolling in a well (without friction). In any potential well we can set up a rolling marble at any energy as determined by the highest points attained.

If E is larger than *any* point on the curve $V(x)$, then dx/dt can *never* vanish, and so an oscillation is impossible. Thus in figure 4 we see that if $E > 1/16$, where $1/16$ is the value of y at the maxima of V , x will actually move to $+\infty$ if $dx/dt(0) > 0$ and to $-\infty$ if $dx/dt(0) < 0$.

Any potential can be analyzed this way. In figure 5 we show an example. There are two small wells, in each of which an oscillation is defined. But also there is a single “big” well with two bumps at the bottom, and for E between the smaller and the large local maxima, the oscillation will involve both of the smaller wells. If E exceeds the larger maxima, x will go to $-\infty$ as $t \rightarrow \infty$. The phase plane corresponding to figure 5 (with $m = 1$), is also shown in figure 5.

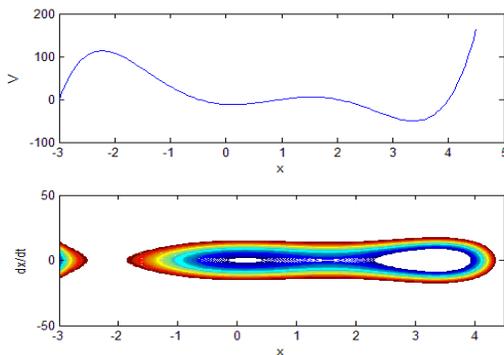


Figure 5. Above, a general potential function, and below the phase plane of the corresponding nonlinear oscillator.

4 An oscillator with friction

The setup of figure 1 was for a simple harmonic oscillator on a slippery surface. In fact friction is present, and it is interesting to see how friction can be modeled in this case.

Classical friction is described by the physical laws of Amontons (1663-1705), a French physicist who studied friction experimentally. Friction takes many forms, but the classical result we shall use is the our block sliding on a surface experiences a force independent of the velocity but proportional to its weight.

Surprisingly, the force is independent of the “contact area”, but this is a misleading statement. In fact the microscopic bumps or *asperities* deform under the weight, so that the actual contact area is indeed proportional to weight.

In any event the experimental result is that

$$F_{friction} = -gm\mu \text{sign}(dx/dt). \quad (27)$$

Here gm is the weight of the body in a gravitational field, and μ is the *coefficient of moving friction*, often about .4, the constant of proportionality in Amontons’ law. We also know that the frictional force acts opposite to the direction of motion. Hence we include the *sign* function which is just the sign of its argument, in this case the velocity. Since the friction force is opposite in sign from that of the velocity, there is a minus sign on the right.

To see what happens with our oscillator when friction is present, suppose that we start by stretching the spring to $x = L > 0$, hold the mass so that $dx/dt(0) = 0$, then release it. Now *if* the mass begins to move when released, it will move to the left ($\text{sign}(dx/dt) = -1$), so that $F_{friction} = gm\mu$. But if this force is greater than the initial pulling force of the spring, kL , clearly the mass will not move at all! So we must assume that $gm\mu < kL$. If this inequality is satisfied the mass can move and the equation of motion is

$$m\frac{d^2x}{dt^2} + kx = gm\mu. \quad (28)$$

Solving this equation with $x(0) = L, dx/dt(0) = 0$, we get

$$x = a + (L - a) \cos \omega t, \quad \omega = \sqrt{\frac{k}{m}}, \quad dx/dt = -\omega(L - a) \sin \omega t, \quad a = gm\mu/k. \quad (29)$$

In the phase plane of $(x, dx/dt)$, the motion is along an arc of the ellipse given by

$$(x - a)^2 + \frac{m}{k}(dx/dt)^2 = (L - a)^2, \quad (30)$$

see figure 6. (Problem: derive the last equation using an energy-type integral, and the initial conditions.) One this path hits the negative x -axis, the sign of the velocity switches to positive, so we now have to solve

$$m\frac{d^2x}{dt^2} + kx = -gm\mu \quad (31)$$

with initial conditions (we will set the clock to 0 so initial conditions are at $t = 0$ for the next arc) given by $x(0) = 2a - L, dx/dt(0) = 0$. The equation for the next arc is then

$$(x + a)^2 + \frac{m}{k}(dx/dt)^2 = (L - 3a)^2, \quad (32)$$

This continues until the path hits the x -axis a distance $\leq a$ from the origin. Then the spring cannot overcome friction and the mass stops. We show an example of this in figure 6.

We remark that we have not considered here that there can be a slight difference between the frictional force which acts on a sliding object, and the force needed to *start* sliding. The latter is usually called the *static friction*, and its coefficient can be slightly greater than the μ we are using here. Whether or not this effect is important depends on the materials. Since our mass is at rest for an instant only during the oscillation, it is reasonable to disregard the rest state as not really “static”.

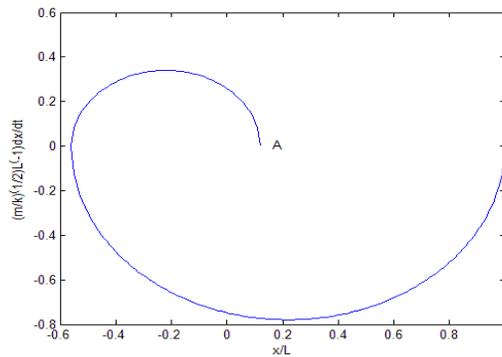


Figure 6. Above, an oscillation with friction, $a/L = .22$. The lower curve is a circular arc of radius $1 - a/l = .78$, centered at $(.22, 0)$. The upper curve is a circular arc of radius $1 - 3a/L = .34$, centered at $(-.22, 0)$. The last intersection with the x -axis (point A), is at $(.12, 0)$. Since $.12 < .22$ the mass comes to rest there.