7/7

Due 3/30/05
Math Modeling

① a) How fast are cars moving before they encounter the wave?

$u = \frac{q}{\rho}$

$q = u_{max}\rho(1-\rho/\rho_{max})$

$= 70\rho(1-\rho/377)$

$\rho_0 = 250$

$q = 70(250)[1-250/377]$

$= 5895.225$

$\frac{q}{\rho} = \frac{5895.225}{250} = u = \boxed{23.58 \text{ mph}}$ ✓

1 b) $q = 70\rho(1-\rho/377)$ $x = -1$ mile at $t = 0$.

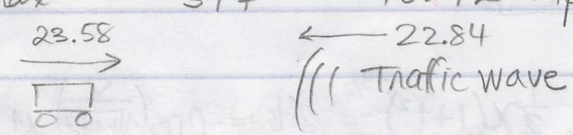
$\frac{dq}{d\rho} = 70 - 140\rho/377$ What time encounter wave?

$= 70 - 140(250)/377$ ✓ $V = \frac{D}{T}$

$= \boxed{-22.84 \text{ mph}}$ ✓ $23.58-(-22.84) = \frac{1}{T}$

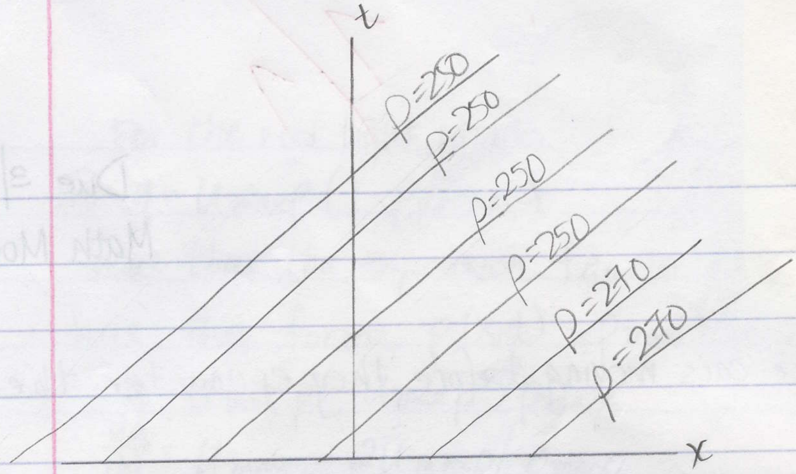
$t = .022$ hr or in $\boxed{1.29 \text{ min}}$

c) $\frac{u_{max}\rho}{\rho_{max}} = \frac{70\cdot 250}{377} = 46.42$ mph



Driver in the pack will see the traffic wave ✓
approaching at $23.58-(-22.84) = \boxed{46.42 \text{ mph}}$
This agrees with the results in (a) and (b).

Show $u(\rho) - V(\rho) = \frac{u_{max}\rho}{\rho_{max}}$



② a) $\frac{\partial f}{\partial t} + \frac{xt}{1+t^2}\frac{\partial f}{\partial x} = 0 \qquad f(x,0) = \sin(x)$

$$\frac{dx}{dt} = \frac{xt}{1+t^2}$$

$$x(1+t^2)dx = xt\,dt$$

$$\int \frac{1}{x}dx = \int \frac{t}{1+t^2}dt$$

$$\ln x = \frac{1}{2}\ln(1+t^2) + \phi$$

$$\phi = \ln x - \frac{1}{2}\ln(1+t^2) = \text{constant}$$

$F(\phi) = f(x,t)$

$$= \ln x - \ln\sqrt{1+t^2}$$

$$= \ln\left(\frac{x}{\sqrt{1+t^2}}\right)$$

e^{ϕ} is still a constant

$$\phi = e^{\phi} = \frac{x}{\sqrt{1+t^2}}$$

$$f(x,t) = F(\phi) = F\left(\frac{x}{\sqrt{1+t^2}}\right)$$

1

$$f(x,0) = \sin(x) = F\left(\frac{x}{1}\right) = F(x) \quad ✓$$

$$\boxed{f(x,t) = \sin\left(\frac{x}{\sqrt{1+t^2}}\right)} \quad ✓$$

Check:

$$\frac{\partial f}{\partial t} = \cos\left(\frac{x}{\sqrt{1+t^2}}\right)\cdot -\frac{1}{\cancel{2}}x(1+t^2)^{-3/2}\cdot\cancel{2}t = -\cos\left(\frac{x}{\sqrt{1+t^2}}\right)xt(1+t^2)^{-3/2}$$

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{\sqrt{1+t^2}}\right)\frac{1}{\sqrt{1+t^2}}$$

$$= -\cos\left(\frac{x}{\sqrt{1+t^2}}\right)xt(1+t^2)^{-3/2} + \frac{xt}{1+t^2}\cos\left(\frac{x}{\sqrt{1+t^2}}\right)\frac{1}{\sqrt{1+t^2}} \quad ✓$$

$$= -\cos\left(\frac{x}{\sqrt{1+t^2}}\right)xt(1+t^2)^{-3/2} + xt(1+t^2)^{-3/2}\cos\left(\frac{x}{\sqrt{1+t^2}}\right)$$

$$= 0 \quad ✓$$

b) $\frac{\partial f}{\partial t} + \frac{1}{1+x}\frac{\partial f}{\partial x} = 0$ $\qquad f(x,0) = \sin(x)$

$$\frac{dx}{dt} = \frac{1}{1+x}$$

$$\int (1+x)\,dx = \int 1\,dt$$

$$x + \frac{x^2}{2} - t = \phi$$

<u>Hint:</u> $F(\phi) = -1 + \sqrt{1+2\phi} = f(x,t)$

$$= -1 + \sqrt{1+2\left(x+\frac{x^2}{2}-t\right)}$$

$$\boxed{f(x,t) = -1 + \sqrt{1+2x+x^2-2t}}$$

1 $\quad f(x,0) = x \longrightarrow$ initial condition

$$= -1 + \sqrt{1+2x+x^2+0}$$

$$= -1 + \sqrt{(x+1)^2}$$

$$= \cancel{-1} + x + \cancel{1} = x \checkmark$$

<u>Check:</u>

$$\frac{\partial f}{\partial t} = \frac{1}{2}(1+2x+x^2-2t)^{-1/2} \cdot -2 = -(1+2x+x^2-2t)^{-1/2}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}}(1+2x+x^2-2t)^{-1/2} \cdot \cancel{2}x+\cancel{2} = x+1(1+2x+x^2-2t)^{-1/2}$$

$$\frac{\partial f}{\partial t} + \frac{1}{1+x}\frac{\partial f}{\partial x} = 0$$

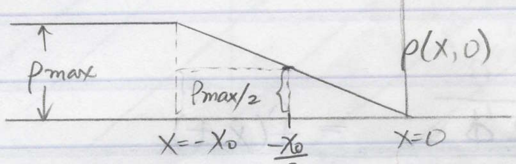
$$= -(1+2x+x^2-2t)^{-1/2} + \cancel{\frac{1}{1+x}(x+1)}(1+2x+x^2-2t)^{-1/2}$$

$$= 0 \checkmark$$

$\checkmark$

③ 71.1

$$q(\rho) = a\rho\left[\ln(\rho_{max}) - \ln\rho\right]$$



Two hours later, where does $\rho = \rho_{max}/2$

$\frac{dq}{d\rho} = a\rho\ln(\rho_{max}) - a\rho\ln\rho$

$= a\ln(\rho_{max}) - a\ln\rho - a$

$= a(\ln(\rho_{max}) - \ln\rho - 1) = a\left[\ln\left(\frac{\rho_{max}}{\rho}\right) - 1\right]$

1

Equation of characteristic

$x = \left(\frac{\partial q}{\partial \rho}\right)t + x_0$

where $t=2$, $\rho = \rho_{max}/2$

$x = a\left[\ln\frac{\rho_{max}}{\rho_{max}/2} - 1\right](2) - \frac{x_0}{2}$

$= a(\ln 2 - 1)(2) - \frac{x_0}{2} = \boxed{2a\ln 2 - 2a - \frac{x_0}{2}}$ ✓

71.2 ④ Referring to theoretical flow-density relationship of #3, show that the density wave velocity relative to a moving car is the same constant no matter what the density.

$\frac{\partial q}{\partial \rho} = \rho\frac{du}{d\rho} + u$ $\qquad q(\rho) = \rho\, u(\rho)$

$\frac{\partial q}{\partial \rho} - u = \rho\frac{du}{d\rho}$ $\qquad u(\rho) = \frac{q(\rho)}{\rho}$

using results of #3, $u(\rho) = a\left[\ln(\rho_{max}) - \ln\rho\right]$

$\frac{\partial q}{\partial \rho} = a\left[\ln\left(\frac{\rho_{max}}{\rho}\right) - 1\right]$

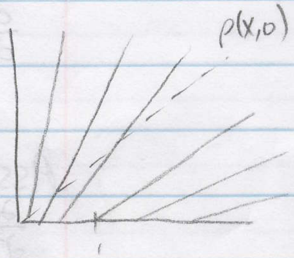
$\frac{\partial q}{\partial \rho} - u = \cancel{a\ln\left(\frac{\rho_{max}}{\rho}\right)} - \cancel{a\ln\frac{\rho_{max}}{\rho}} - a$

$\boxed{\frac{\partial q}{\partial \rho} - u = -a = \rho\frac{du}{d\rho}}$ $\quad a =$ given constant.

This shows that no matter what the density, density wave velocity relative to u is constant.

⑤ $\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0, \quad x > 0$ $\qquad \rho(x,0) = x$

$\rho_0(x_0) = x_0$



$x = \rho^2(x_0, 0)t + x_0$

$x = x_0^2 t + x_0$ → Solve for $x_0(x,t)$

1

$x_0^2 t + x_0 - x = 0$ $\qquad x_0 = \frac{-1 \pm \sqrt{1+4tx}}{2t}$ ⇒ don't need <0

$x_0 = \rho(x,t) = \frac{-1+\sqrt{1+4xt}}{2t}$ ✓

use int. cond. $\rho(x,0) = \frac{-1+\sqrt{1+4xt}}{2t} = \frac{0}{0}$ Use L'Hopital's Rule ✓

0.5 $\lim_{(x,t)\to(x,0)} \frac{\frac{1}{2}(1+4xt)^{-1/2} \cdot 4x}{2} = \frac{\frac{1}{2}(1) \cdot 4x}{2} = x$ ✓

Check: $\rho(x,t) = \frac{1}{2}t^{-1}\left(-1 + (1+4xt)^{1/2}\right)$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2}t^{-2}\left(-1+\sqrt{1+4xt}\right) + \frac{1}{2t}\left(\frac{1}{2}(1+4xt)^{-1/2}\right) \cdot 4x$$

$$= -\frac{1}{2}t^{-2}\left(-1+\sqrt{1+4xt}\right) + \frac{x}{t}\frac{1}{\sqrt{1+4xt}}$$

$$\frac{\partial \rho}{\partial x} = \frac{1}{4t}(1+4xt)^{-1/2} \cdot 4t$$

$$= (1+4xt)^{-1/2}$$

0.5

$$\rho^2 = \left(-\frac{1}{2t} + \frac{\sqrt{1+4xt}}{2t}\right)\left(-\frac{1}{2t} + \frac{\sqrt{1+4xt}}{2t}\right)$$

$$= \frac{1}{4t^2} - \frac{2\sqrt{1+4xt}}{4t^2} + \frac{1+4xt}{4t^2}$$

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0$$

$$= -\frac{1}{2}t^{-2}\left(-1+\sqrt{1+4xt}\right) + \frac{x}{t}\frac{1}{\sqrt{1+4xt}} + \frac{2+4xt-2\sqrt{1+4xt}}{4t^2}\left(\frac{1}{\sqrt{1+4xt}}\right)$$ ✓

$= 0$ ✓

For the red light problem with

⑥ $q = u_{max}\rho(1-\rho/\rho_{max})$

Show that the expansion fan in the transition region has the form $\rho(x,t) = \rho_{max}\left(\frac{u_{max}t - x}{2u_{max}t}\right)$

$$q = u_{max}\rho - u_{max}\rho^2/\rho_{max}$$

$$\frac{dq}{d\rho} = u_{max} - 2u_{max}\rho/\rho_{max}$$

Assume $\rho(x,t) = R(x/t) = R(y)$

$$\frac{\partial\rho}{\partial t} + (u_{max} - 2u_{max}\rho/\rho_{max})\frac{\partial\rho}{\partial x} = 0$$

$$\frac{\partial R}{\partial t} = \frac{dR}{dy}\frac{\partial y}{\partial t} = \frac{dR}{dy}\left(-\frac{x}{t^2}\right)$$

$$\frac{\partial R}{\partial x} = \frac{dR}{dy}\frac{\partial y}{\partial x} = \frac{dR}{dy}\left(\frac{1}{t}\right)$$

$$-\frac{y}{t}\left(\frac{dR}{dy}\right) + (u_{max} - 2u_{max}R/\rho_{max})\frac{dR}{dy}\frac{1}{t} = 0$$

1

$$-y + u_{max} - \frac{2u_{max}R}{\rho_{max}} = 0$$

$$-\frac{2u_{max}R}{\rho_{max}} = y - u_{max}$$

$$2u_{max}R = -y\rho_{max} + u_{max}\rho_{max}$$

$$R = \rho_{max}\left(\frac{-y + u_{max}}{2u_{max}}\right)$$

$$y = \frac{x}{t}$$

$$= \rho_{max}\left(\frac{-\frac{x}{t} + u_{max}}{2u_{max}}\right)$$

$$R(x/t) = \rho(x,t) = \rho_{max}\left(\frac{u_{max} - x}{2u_{max}t}\right)$$