Problem Set #7

7.5/8

1) 2 Worker Bucket Brigade: worker 1: 3 hours
worker 2: 1 hour

$v_1 = 1/3$ $\quad t = \frac{d}{v}$ $\quad x_2^{(1)} = 1/2$ initial reset
$v_2 = 1$

Subsequent Resets (k=2,3,4): $x_2^{(k+1)} = \frac{v_1}{v_2}(1 - x_2^{(k)})$
$= 1/3(1 - x_2^{(k)})$

$x_2^{(2)} = \frac{1}{3}(1 - 1/2) = 1/6$

$t = 1/2 / 1$
$= 1/2 + 0 = 1/2$

$k_2 : (1/6, 1/2)$

$x_2^{(3)} = 1/3(1 - 1/6) = \frac{5}{18}$

$t = \frac{1}{2} + 5/6 / 1 = 4/3$

$k_3 : (\frac{5}{18}, \frac{4}{3})$

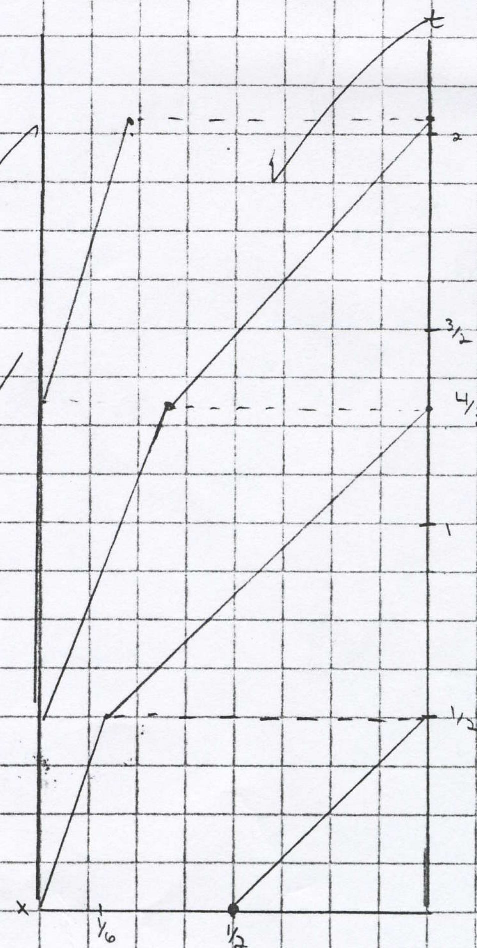
$x_2^{(4)} = \frac{1}{3}(1 - \frac{5}{18}) = \frac{13}{54}$

$t = \frac{4}{3} + 13/18 / 1 = \frac{37}{18}$

$k_4 : (\frac{13}{54}, \frac{37}{18})$

Summary of Resets: x-t graph:

$k_1 : (1/2, 0)$
$k_2 : (1/6, 1/2)$
$k_3 : (5/18, 4/3)$
$k_4 : (13/54, 37/18)$



Worker Order Switched: $v_1 = 1$ $x_2 = 1/2$
$v_2 = 1/3$

$x_2^{(2)} = 3(1 - 1/2) = 3/2$ $t = 1/2 / 1/3 + 0 = \frac{3}{2}$

$\hookrightarrow$ blocking ocurred

speed now depends entirely on worker 2's velocity.

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since $v_2 = 1/3$ it takes 3 hrs to move 1 unit and reset always occurs at 0 due to blocking

since two widgets are produced every 3 hrs, the sub-obtimal production rate is obtained ✓

$P_s = 2v_2 = 2/3 = 2/3$ ✓

✓

3) When the equilibrium is obtained, the points (x_2, t) are always:

$$\left(\frac{1}{4}, \frac{3}{4} + \sum_{i=1}^{k-1} t_i\right)$$

Thus each worker always spends 3/4 time working on each widget as can be shown by their respective velocities and distinces

0.5

$w_2: \quad t = \frac{d}{v} = \frac{3/4}{1} = \frac{3}{4}$ ✓

$w_2: \quad t = \frac{d}{v} = \frac{1/4}{1/3} = 3/4$

0.5 This equipartitioning of time is produced in any balanced equilibrium set due to the nature of the velocities and distance relationship. yes but do it explicitly...

This time is simply the time to reset and is given by $\boxed{t = \frac{1 - x_2^{(k)}}{v_2}}$ (In this case $t_r = \frac{1 - 1/4}{1} = \frac{3}{4}$) □

1) cont Difference Equation for $x_2^{(k)}$:

$$x_2^{(k+1)} = \frac{1}{3}(1 - x_2^{(k)})$$
$$3x_2^{(k+1)} = 1 - x_2^{(k)}$$

$$3x_2^{(k+1)} + x_2^{(k)} = 1$$

Solution of the Difference Equation

trial solution: $A + B\lambda^n = x_2^{(k)}$ ✓

$$3(A + B\lambda^{k+1}) + (A + B\lambda^k) = 1$$

$3A + A = 1$ ✓
$A = 1/4$ ✓

$3B\lambda^{k+1} + B\lambda^k = 0$
$3\lambda + 1 = 0$
$\lambda = -1/3$ ✓

$x_2^{(k)} = \frac{1}{4} + B(-1/3)^k$
$x_2^{(1)} = \frac{1}{2} = \frac{1}{4} + B(-\frac{1}{3})$
$B = -3/4$ ✓

Solution: $x_2^{(k)} = \frac{1}{4} - \frac{3}{4}(-1/3)^k$

as $k \to \infty$ $x_2^{(k)} \to \frac{1}{4}$

1 $\therefore$ a balanced equilibrium is reached w/ $x_2^{(1)} = \frac{1}{4}$

Production Rate Comparison

Optimal: $P = v_1 + v_2 = 4/3$ ✓

When $x_2^{(1)} = 1/4$

$x_2^{(2)} = \frac{1}{3}(1 - \frac{1}{4}) = \frac{1}{4}$

$t = \frac{3/4}{1} + 0$
$t = 3/4$

$k_2 : (\frac{1}{4}, \frac{3}{4})$

since time $= \frac{3}{4}$ $\Rightarrow$ 1 widget produced every $3/4$ time unit
$\Rightarrow$ $4/3$ widgets produced every time unit

$4/3 = 4/3$ ✓ $\Rightarrow$ optimal production rate is obtained ✓

4) Three-Worker Production Line

$v_1 = 2$ $\qquad x_2^{(1)} = 1/4$
$v_2 = 1$ $\qquad x_3^{(1)} = 2/3$
$v_3 = 3$

$$x_3^{(k+1)} = \min\left[x_2^{(k)} + \frac{v_2}{v_3}\left(1 - x_3^{(k)}\right), 1\right]$$

$$x_2^{(k+1)} = \min\left[\frac{v_1}{v_3}\left(1 - x_3^{(k)}\right), x_3^{(k+1)}\right]$$

0.5 For calculations and proof of convergence, see attached code and output. → when using a computer, you should ~~always~~ print out the a graph.

To demonstrate that I can do this without the aid of a computer: → a computer is good enough!

$$x_3^{(k+1)} = \min\left[\frac{1}{4} + \frac{1}{3}\left(1 - \frac{2}{3}\right), 1\right]$$

$$= \min\left[\frac{13}{36}, 1\right]$$

$$= \frac{13}{36}$$

$$x_2^{(k+1)} = \min\left[\frac{2}{3}\left(1 - \frac{2}{3}\right), \frac{13}{36}\right]$$

$$= \min\left[\frac{2}{9}, \frac{13}{36}\right]$$

$$= \frac{2}{9}$$

repeated iterations of this algorithm demonstrate that (x_2, x_3) converge to $(1/3, 1/2)$

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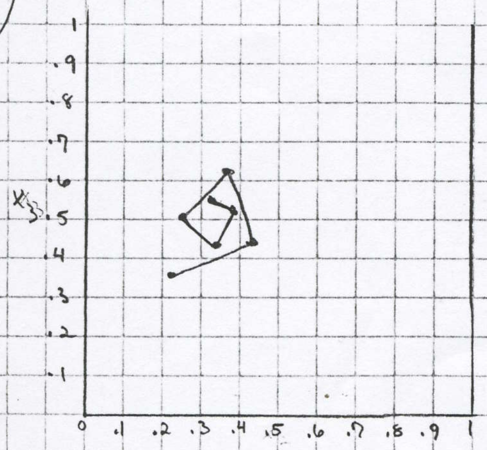
3 Worker BB



Production Rate Comparison

$$P_0 = v_1 + v_2 + v_3$$
$$= 2 + 1 + 3 = 6$$

Since No Blocking Occurs:
Rate is governed by time to reset v_3: $\dfrac{1 - x_3^{(k)}}{v_3}$

$$= \frac{1 - 1/2}{3} = \frac{1}{6}$$

∴ 1 widget produced every 1/6 time unit
⇒ 6 widgets produced every time unit
Optimal Rate is obtained

i) Solving in terms of $x_2^{(k)}$

$$x_2^{(k+1)} = \frac{2}{3}(1 - x_3^{(k)})$$

$$3x_2^{(k+1)} = 2 - 2x_3^{(k)}$$

$$\frac{2 - 3x_2^{(k+1)}}{2} = x_3^{(k)} \qquad \frac{2 - 3x_2^{(k+2)}}{2} = x_3^{(k+1)}$$

$$x_3^{(k+1)} = x_2^{(k)} + \frac{1}{3}(1 - x_3^{(k)})$$

1

$$\frac{2 - 3x_2^{(k+2)}}{2} = x_2^{(k)} + \frac{1}{3}\left(1 - \frac{2 - 3x_2^{(k+1)}}{2}\right)$$

$$6 - 9x_2^{(k+2)} = 6x_2^{(k)} + 2 - 2 + 3x_2^{(k+1)}$$

$$* \quad 3x_2^{(k+2)} + x_2^{(k+1)} + 2x_2^{(k)} = 2$$

Trial Solution : $A + B\lambda^k = x_2^{(k)}$

$$3(A + B\lambda^{k+2}) + (A + B\lambda^{k+1}) + 2(A + B\lambda^k) = 2$$

$$3A + A + 2A = 2 \qquad 3B\lambda^{k+2} + B\lambda^{k+1} + 2B\lambda^k = 0$$

$$A = 1/3 \qquad 3\lambda^2 + \lambda + 2 = 0$$

1

$$\lambda = \frac{-1 \pm \sqrt{1 - 24}}{6} = \frac{-1 \pm i\sqrt{23}}{6}$$

$$\Rightarrow x_2^{(k)} = \frac{1}{3} + B\left(\frac{-1 + i\sqrt{23}}{6}\right)^k$$

as $k \to \infty \quad x_2^{(k)} \to 1/3 \quad \square$

continued on the back

ii) Solving in terms of $X_3^{(k)}$

$$X_3^{(k+1)} = X_2^{(k)} + \frac{1}{3}(1 - X_3^{(k)})$$

$$X_2^{(k)} = X_3^{(k+1)} - \frac{1}{3}(1 - X_3^{(k)}) \qquad X_2^{(k+1)} = X_3^{(k+2)} - \frac{1}{3}(1 - X_3^{(k+1)})$$

$$X_2^{(k+1)} = \frac{2}{3}(1 - X_3^{(k)})$$

$$X_3^{(k+2)} - \frac{1}{3}(1 - X_3^{(k+1)}) = \frac{2}{3}(1 - X_3^{(k)})$$

$$3X_3^{(k+2)} - 1 + X_3^{(k+1)} = 2 - 2X_3^{(k)}$$

$$3X_3^{(k+2)} + X_3^{(k+1)} + 2X_3^{(k)} = 3$$

trial solution: $x_3^{(k)} = A + B\lambda^k$

$$3(A + B\lambda^{k+2}) + (A + B\lambda^{k+1}) + 2(A + B\lambda^k) = 3$$

$3A + A + 2A = 3$

$A = \frac{1}{2}$

$$3B\lambda^{k+2} + B\lambda^{k+1} + 2B\lambda^k = 0$$

$$3\lambda^2 + \lambda + 2 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-24}}{6} = \frac{-1 \pm i\sqrt{23}}{6}$$

$$X_3^{(k)} = \frac{1}{2} + B\left(\frac{-1 \pm i\sqrt{23}}{6}\right)^k$$

as $k \to \infty$ $\quad X_3 \to \frac{1}{2}$

$\therefore$ (X_2, X_3) converge to $(\frac{1}{3}, \frac{1}{2})$

This is the equilibrium of the system $\square$