1. Consider a fluid of constant density in two dimensions with gravity, and suppose that the vorticity $u_y - v_x$ is everywhere constant and equal to $\omega$. Show that the velocity field has the form $(u, v) = (\phi_x + \chi_y, \phi_y - \chi_x)$ where $\phi$ is harmonic and $\chi$ is any function of $x, y$ (independent of $t$), satisfying $\nabla^2 \chi = -\omega$. Show further that
\[
\nabla(\phi_t + \frac{1}{2}q^2 + \omega \psi + p/\rho + gz) = 0
\]
where where $\psi$ is the streamfunction for $\vec{u}$, i.e. $\vec{u} = (\psi_y, -\psi_x)$, and $q^2 = u^2 + v^2$.

2. Show that, for an incompressible fluid, but one where the density can vary independently of pressure (e.g. salty seawater), the vorticity equation is
\[
\frac{D\omega}{Dt} = \omega \cdot \nabla \vec{u} + \rho^{-2} \nabla \rho \times \nabla p.
\]
Interpret the last term on the right physically. (e.g. what happens if lines of constant $p$ are $y = \text{constant}$ and lines of constant $\rho$ are $x - y = \text{constant}$?). Try to understand how the term acts as a source of vorticity, i.e. causes vorticity to be created in the flow.

3. For steady two-dimensional flow of a fluid of constant density, we have
\[
\rho \vec{u} \cdot \nabla \vec{u} + \nabla p = 0, \nabla \cdot \vec{u} = 0.
\]
Show that, if $\vec{u} = (\psi_y, -\psi_x)$, these equations imply
\[
\nabla \psi \times \nabla (\nabla^2 \psi) = 0.
\]
Thus, show that a solution is obtained by giving a function $H(\psi)$ and then solving $\nabla^2 \psi = H'(\psi)$. Show also that the pressure is given by $\frac{p}{\rho} = H(\psi) - \frac{1}{2}(\nabla \psi)^2 + \text{constant}$.

4. Prove Ertel’s theorem for a fluid of constant density: If $f$ is a scalar material invariant, i.e. $Df/Dt = 0$, then $\omega \cdot \nabla f$ is also a material invariant, where $\omega = \nabla \times \vec{u}$ is the vorticity field.

5. A steady Beltrami flow is a velocity field $\vec{u}(\vec{x})$ for which the vorticity is always parallel to the velocity, i.e. $\nabla \times \vec{u} = f(\vec{x})\vec{u}$ for some scalar function $f$. Show that if a steady Beltrami field is also the velocity field of an inviscid fluid of constant density, the necessarily $f$ is constant on streamlines. What is the corresponding pressure? Show that $\vec{u} = (B \sin y + C \cos z, C \sin z + A \cos x, A \sin x + B \cos y)$ is such a Beltrami field with $f = -1$. (This last flow an example of a velocity field yielding chaotic particle paths. This is typical of 3D Beltrami flows with constant $f$, according to a theorem of V. Arnold.)