Fluid Dynamics I PROBLEM SET 8 Due by the last class (December 12).

1. Consider the uniform slow motion with speed U of a viscous fluid past a spherical bubble of radius a, filled with air. Do this by modifying the Stokes flow analysis for a rigid sphere as follows. The no slip condition is to be replaced on $r = a$ by the condition that both u_r and the tangential stress $\sigma_{r\theta}$ vanish. (This latter condition applies since there is no fluid within the bubble to support this stress.) Show in particular that

$$\Psi = \frac{U}{2}(r^2 - ar)\sin^2\theta$$

and that the drag on the bubble is $D = 4\pi\mu U a$. Note: On page 235 of Batchelor see the analysis for a bubble filled with a second liquid of viscosity $\bar{\mu}$. The present problem is for $\bar{\mu} = 0$.

2. Consider two-dimensional Stokes flow past a circular cylinder of radius a. Show that the problem reduces to the biharmonic equation for the two-dimensional stream function ψ,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \psi = 0,$$

with conditions $\partial\psi/\partial r = \partial\psi/\partial\theta = 0$ on $r = a$ and $\psi \sim Ur\sin\theta$ as $r \to \infty$. Seeking a solution of the form $f(r)\sin\theta$, show that this leads to

$$f = Ar^3 + Br\log r + Cr + D/r$$

and hence that there is no solution of the required form. This is *Stokes' Paradox*, as discussed in class.

3. Prove that Stokes flow past a given, rigid body is unique, as follows. Show if $p_1, \mathbf{u}_1$ and $p_2, \mathbf{u}_2$ are two solutions of

$$\nabla p - \mu\nabla^2\mathbf{u} = 0, \nabla\cdot\mathbf{u} = 0,$$

satisfying $u_i = -U_i$ on the body and

$$\mathbf{u} \sim= O(1/r), \frac{\partial u_i}{\partial x_j}, p \sim O(1/r^2)$$

as $r \to \infty$, then the two solutions must agree. (Hint: Consider the integral of $\partial/\partial x_i(w_j\partial w_j/\partial x_i)$ over the region exterior to the body, where $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$.)

4. Two small spheres of radius a and density ρ_s are falling in a viscous fluid with centers at P and Q. The line PQ has length $L \gg a$ and is perpendicular to gravity. Using the Stokeslet approximation to the Stokes solution past a sphere, and assuming that each sphere sees the unperturbed Stokes flow of the other sphere, show that the spheres fall with the same speed

$$U \approx U_s(1 + ka/L + O(a^2/L^2)),$$

and determine the number k. Here $U_s = 2a^2g/9\nu(\rho_s/\rho - 1)$ is the settling speed of a single sphere in Stokes flow.

5. *Oseen's equations* are sometimes proposed as a model of the Navier-Stokes, equations, in the study of steady viscous flow past a body. Oseen's equations, for a flow with velocity $(U, 0, 0)$ at infinity, are

$$U\frac{\partial\mathbf{u}}{\partial x} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{u} = 0, \nabla\cdot\mathbf{u} = 0.$$

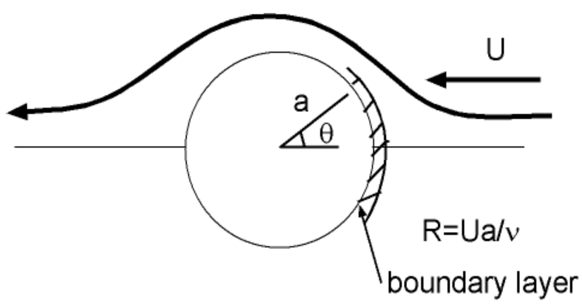
(a) For the Oseen model, and for a flat plate aligned with the flow, carry out Prandtl's simplifications for deriving the boundary-layer equations in two dimensions, given that the boundary condition of no slip is retained at the body. That is find the form of the boundary layer on a flat plate of length L aligned with the flow at infinity, according to Oseen's model, and show that in the boundary layer the the x-component of velocity, u, satisfies

$$U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial y^2}.$$

What are the boundary conditions on u for the flat-plate problem? Find the solution, by assuming that u is a function of $y\sqrt{\frac{U}{\nu x}}$, for $0 < x < L$.

(b) Compute the drag coefficient of the plate (drag divided by $\rho U^2 L$, and remember there are two sides), in the Oseen model.

6. What are the boundary-layer equations for the boundary-layer on the front portion of a circular cylinder orf radius a, when the free stream velocity is $(-U, 0, 0)$ (see figure 1)? (Use cylindrical polar coordinates). What is the role of the pressure in the problem? Be sure to include the effect of the pressure as an explicit function in your momentum equation, the latter being determined by the potential flow past a circular cylinder studied previously. Show that, by defining $x = a\theta, \bar{y} = (r - a)\sqrt{R}$ in the derivation of the boundary-layer equations, the equations are equivalent to a boundary layer on a flat plate aligned with the free stream, in rectangular coordinates, but with pressure a given function of x.



7. For a *cylindrical* jet emerging from a hole in a plane wall, we have a problem analogous to the 2D jet considered in class (see figure 2). Consider only the boundary-layer limit. (a) Show that

$$\frac{\partial}{\partial z}(u_z^2)+\frac{1}{r}\frac{\partial}{\partial r}(ru_r u_z)-\frac{\nu}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r})=0,$$

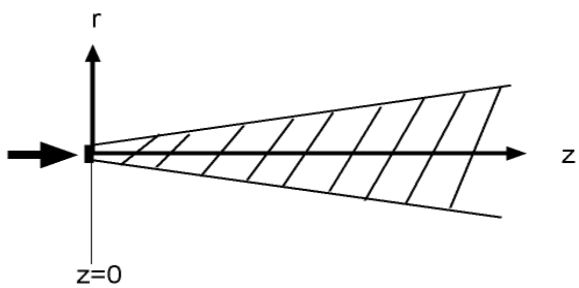
and hence that the momentum M is a constant, where

$$M=2\pi\rho\int_0^\infty ru_z^2dr.$$

(b) Letting $(u_z,u_r)=(1/r)(\psi_r,-\psi_z)$ where $\psi(0,z)=0$ show that we must have $\psi=zf(\eta),\ \eta=r^2/z^2$. Determine the equation for f and thus show that the boundary-layer limit has the form

$$f=4\nu\frac{\eta}{\eta+\eta_0},$$

where η_0 is a constant. Express η_0 in terms of M, the momentum flux of the jet defined above.



8. When as one moves downstream within a boundaary layer, the pressure increases, we say that we have an *adverse pressure gradient*. With an adverse pressure gradient separation can occur, or else the boundary layer problem may not have a solution. The latter case is illustrated by the present problem.

Consider the Prandtl boundary-layer equations with $U(x)=1/x$, so $p(x)/\rho=p_\infty-1/(2x^2)$. Verify that the similarity solution has the form $\psi=f(\eta),\eta=y/x$. Find the equation for f. Show that there is no continuously differentiable solution of the equation which satisfies $f(0)=f'(0)=0$ and $f'\to 1, f''\to 0$ as $\eta\to\infty$. (Hint: Obtain an equation for $g=f'$.)