6. This problem will study flow past a slender axisymmetric body whose surface is given (in cylindrical polar coordinates), by $r = R(z), 0 \leq z \leq L$. Here $R(z)$ is continuous, and positive except at $0, L$ where it vanishes. By "slender" we mean that $\max_{0 \leq x \leq L} R << L$. The body is placed in the uniform flow $(u_z, u_r, u_\theta) = (U, 0, 0)$. We are interested in the steady, axisymmetric potential flow past the body. It can be shown that such a body perturbs the free stream by only a small amount, so that in particular, $u_z \approx U$ everywhere. On the other hand the flow must be tangent at the body, which implies $\phi_r(z, R(z)) \approx UdR/dz, 0 < z < L$.

We look for a representation of ϕ as a distribution of sources with strength $f(z)$. Thus

$$\phi(z,r) = -\frac{1}{4\pi}\int_0^L \frac{f(\zeta)}{\sqrt{(z-\zeta)^2 + r^2}} d\zeta.$$

(a) Compute $\frac{\partial \phi}{\partial r}$, and investigate the resulting integral as $r \to 0, 0 < z < L$. Argue that the dominant contribution comes near $\zeta = z$, and hence show that $\frac{\partial \phi}{\partial r} \approx \frac{1}{4\pi}\frac{f(z)}{r}\int_{-\infty}^{+\infty}(1+s^2)^{-3/2} ds$ for $r << L$.

(b) From the above tangency condition, deduce that $f(z) \approx dA/dz$ where $A(z) = \pi R^2$ is the cross-sectional area of the body.

(c) By expanding the above expression for ϕ for large z, r, show that in the neighborhood of infinity

$$\phi \approx \frac{1}{4\pi}\frac{z}{(z^2+r^2)^{3/2}}\int_0^L A(\zeta) d\zeta, z^2 + r^2 \to \infty.$$

7. Compute, using the Blasius formula, the force exerted by a simple source at a point $c = be^{i\theta}$ on a circular cylinder of radius $a < b$. Recall that the complex potential for a simple source at c is $w = (Q/2\pi)\log(z - c)$.(Hint: Use the circle theorem, then compute the force from the residue at c.) Verify that the cylinder is pulled toward the source.