

1. (Reading: Batchelor 543-545). Consider axisymmetric motion, with velocity  $\mathbf{u} = (u_r, u_\theta, u_z)$ , of a fluid of constant density. The equation of continuity in cylindrical polar coordinates is

$$\frac{\partial u_z}{\partial z} + \frac{1}{r} \frac{\partial r u_r}{\partial r} = 0.$$

(a) Show that this equations is satisfied if

$$u_z = \frac{1}{r} \frac{\partial \psi}{\partial r}, u_r = -\frac{1}{r} \frac{\partial \psi}{\partial z},$$

for some function  $\psi$ , and verify that

$$\omega_\theta = -\frac{1}{r} L(\psi), L = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}.$$

(b) Assume the flow is *steady*. i.e.  $\partial \mathbf{u} / \partial t = 0$ . Show that if

$$\mathbf{u} \cdot \nabla Q = u_r \frac{\partial Q}{\partial r} + u_z \frac{\partial Q}{\partial z} = 0$$

Then  $Q$  is a function of  $\psi$  alone,  $Q = F(\psi)$ .

(c) Applied to Bernoulli's expression

$$\mathbf{u} \cdot \nabla \left( \frac{p}{\rho} + \frac{1}{2} q^2 \right) = 0$$

show that

$$\frac{p}{\rho} + \frac{1}{2} (u_r^2 + u_\theta^2 + u_z^2) = H(\psi)$$

for some function  $H$ . Applied to the equation for  $u_\theta$ , show that  $ru_\theta = C(\psi)$  for some function  $C$ . Show that this is a special case of Kelvin's theorem.

2. Continuing problem 1, we showed in class that

$$(u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z})(\omega_\theta / r) = \frac{2}{r^2} u_\theta \frac{\partial u_\theta}{\partial z}.$$

From this show (using results of problem 1) that  $\psi$  satisfies

$$L(\psi) = -CC_\psi + r^2 f(\psi)$$

for some functions  $C, f$ . Finally, show from the momentum equation that  $f = H_\psi$  where  $H$  is defined in problem 1.

3. (Reading: Milne-Thomson p. 89, Batchelor p. 384) (a) Prove Kelvin's minimum energy theorem: In a simply-connected domain  $V$  let  $\mathbf{u} = \nabla \phi$ ,  $\nabla^2 \phi = 0$ , with  $\partial \phi / \partial n = f$  on the boundary  $S$  of  $V$ . (This  $\mathbf{u}$  is unique in a simply-connected domain). If  $\mathbf{v}$  is any differentiable vector field satisfying  $\nabla \cdot \mathbf{v} = 0$  in  $V$  and  $\mathbf{v} \cdot \mathbf{n} = f$  on  $S$ , then

$$\int_V |\mathbf{v}|^2 dV \geq \int_V |\mathbf{u}|^2 dV.$$

(Hint: Let  $\mathbf{v} = \mathbf{u} + \mathbf{w}$ , and apply the divergence theorem to the cross term.)