1. We asserted in class that \( \phi = \frac{\mu}{2}(\partial u_i/\partial x_j - \partial u_j/\partial x_i)^2 - \frac{2\mu}{\rho}(\nabla \cdot \vec{u})^2 \) is non-negative. Prove this.

2. Show that for a perfect gas, another form of the energy equation for a viscous, heat conducting fluid is

\[
\rho c_v \frac{DT}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt} = \phi + \nabla \cdot k \nabla T.
\]

(Hint: Start with \( dS = (\frac{\partial S}{\partial T})_v dT + (\frac{\partial S}{\partial v})_T dv \).)

3. Show that for an inviscid gas with zero heat conductivity, (i.e. \( \mu = k = 0 \)), the energy equation may be written

\[
\frac{\partial pe}{\partial t} + \nabla \cdot (\vec{u} pe) = -p \nabla \cdot \vec{u}.
\]

4. (a) Show that

\[
c_p - c_v = T \left( \frac{\partial p}{\partial T} \right)_v \left( \frac{\partial v}{\partial T} \right)_p
\]

(Hint: Regarding \( s \) as a function of \( T, v, ds = (\frac{\partial s}{\partial T})_v dT + (\frac{\partial s}{\partial v})_T dv \), from which we can get an expression for \( (\frac{\partial s}{\partial T})_p \). Now use \( T(\frac{\partial s}{\partial T})_p = c_p, T(\frac{\partial s}{\partial v})_v = c_v \), and a Maxwell relation.) Show that, for a perfect gas, this relation gives \( R = c_p - c_v \).

(b) Show that for a perfect gas

\[
\left( \frac{\partial p}{\partial \rho} \right)_s = \gamma RT.
\]

Note that this quantity equals \( c^2 \) where \( c \) is the speed of sound under isentropic conditions.

5. For the study of thermal convection in water, it is usually assumed that the fluid density is a function of temperature alone. The energy equation is then usually approximated as a temperature equation of the form

\[
\rho c_p \frac{DT}{Dt} - \nabla \cdot k \nabla T = 0.
\]

Justify this as an approximation to the energy equation given in class, namely

\[
\rho c_p DT/Dt - \rho T(\partial v/\partial T)_p Dp/Dt = \phi + \nabla \cdot k \nabla T.
\]

You should make use of the data for water at 20°C given on pages 596 and 597 of Batchelor. (Note \( \beta = v^{-1}(\partial v/\partial T)_p \) and our \( k \) is the same as Batchelor’s \( k_H \).) Assume a characteristic fluid speed is \( U \approx 1 \text{ cm/sec} \), in a fluid layer of thickness \( L \approx 1 \text{ cm} \). That is, use \( U, L \) to estimate terms involving spatial derivatives and velocity. Also take \( L/U \) as a characteristic timescale, one dyne/cm² as a characteristic pressure, and 10°C as typical of \( T \) and its variations. (Note 1 joule = 10⁷ dyne-cm.) Note that the heat conduction term is retained, even though relatively small, because it can be important in boundary layers.

6. For a perfect gas, \( c_v, c_p \) are functions of \( T \) alone and so \( e = \int c_v dT \). Also then \( c_p - c_v = R = \text{constant} \). Using these facts show that for steady flow of a perfect gas Bernoulli’s theorem may be written

\[
\frac{1}{2}u^2 + \int cp(T) dT + \Psi = \text{constant on streamlines}.
\]