Chapter 6

Viscosity and the Navier-Stokes equations

6.1 The Newtonian stress tensor

Generally real fluids are not inviscid or ideal. [1] Modifications of Euler's equations, needed to account for real fluid effects at the continuum level, introduce additional forces in the momentum balance equations. There exists a great variety of real fluids which can be treated at the continuum level, differing in what we shall call their *rheology*. Basically the problem is to identify the forces experienced by a fluid parcel as it is moved about and deformed according to the mathematical description we have developed. Because of the molecular structure of various fluid materials, the nature of these forces can vary considerably and there are many rheological models which attempt to capture the observed properties of fluids under deformation.

The simplest of these rheologies is the *Newtonian viscous fluid*. To understand the assumptions let us restrict attention to the determination of a viscous stress tensor at $\mathbf{x}, t$, which depends only upon the fluid properties within a fluid parcel at that point and time. One could of course imagine fluids where some local average over space determines stress at a point. Also it is easy enough to find fluids with a memory, where the stress at a particular time depends upon the stress history at the point in question.

It is reasonable to assume that the forces dues to the rheology of the fluid are developed by the deformation of fluid parcels, and hence could be determined by the velocity field. If we allow only point properties, deformation of parcels must involve more than just the velocity itself; first and higher-order partial derivatives with respect to the spatial coordinates could be important. (The time derivative of velocity has already been taken into consideration in the acceleration terms.) A moment's thought shows viscous forces cannot depend

[1] In quantum mechanics the superfluid is in many respects an ideal fluid, but the laws governing vorticity, for example, need to be modified.

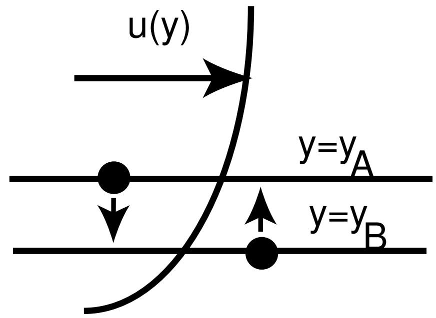

Figure 6.1: Momentum exchange by molecules between lamina in a shear flow.

on velocity. The bulk translation of the fluid with constant velocity produces no force. Thus the deformation of a small fluid parcel must be responsible for the viscous force, and the dominant measure of this deformation should come from the first derivatives of the velocity field, i.e. from the components of the velocity derivative matrix $\frac{\partial u_i}{\partial x_j}$. The Newtonian viscous fluid is one where the stress tensor is *linear* in the components of the velocity derivative matrix, with a stress tensor whose specific form will depend on other physical conditions.

To see why a linear relation of this might capture the dominant rheology of many fluids consider a flow $(u, v) = (u(y), 0)$.Each different plane or *lamina* of fluid, $y=$ constant, moves with a particular velocity. Now consider the two lamina $y = y_A, y_B$ as shown in figure 6.1, moving at velocities $u_B < u_A$. If a molecule moves from B to A, then it is moving from an environment with velocity u_B to an environment with a larger velocity u_A. Consequently it must be accelerated to match the new velcity. According to Newton, a force is therefore applied to the lamina $y = y_A$ in the direction of *negative* x. Similarly a molecule moving from y_A to y_B must slow down, exerting a force on lamina $y = y_B$ in the direction of *positive* x. Thus these exchanges of molecules would tend to reduce the velocity difference between the two lamina. [2]

This tendency to reduce the difference in velocities can be thought of as a force applied to each lamina. Thus if we insert a virtual surface at some position y, a force should be exerted on the surface, in the positive x direction if $du/dy(y) > 0$. Generally we expect the gradients of the velocity components to vary on a length scale L comparable to some macroscopic scale- the size of a container, the size of a body around which the fluid flows, etc. On the other hand the scale of the molecular events envisaged above is very small compared to the macroscopic scale. Thus it is reasonable to assume that the force on the

[2]Perhaps a more direct analogy would be two boats gliding along on the water on parallel paths, one moving faster than the other. If, at the instant they are side by side, an occupant of the fast boat jumps into the slow boat, the slower boat with speed up, and similarly an occupant of the slow boat can slow up the fast boat by jumping into it.

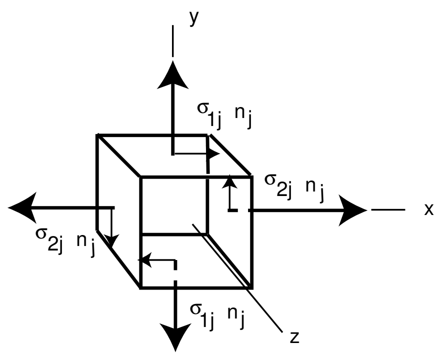

Figure 6.2: Showing why $\sigma_{12} = \sigma_{21}$. The forces are per unit area. The area of each face is Δ^2.

lamina is dominated by the first derivative,

$$F(y) = \mu \frac{du}{dy}. \tag{6.1}$$

The constant of proportionality, μ, is called the *viscosity*, and a fluid obeying this law is called a *Newtonian viscous fluid*.

We have considered so far only a simple planar flow $(u(y), 0)$. In general all of the components of the velocity derivative matrix need to be considered in the construction of the viscous stress tensor. Let us write

$$\sigma_{ij} = -p\delta_{ij} + d_{ij}. \tag{6.2}$$

That is, we have simply split off the pressure contribution and exhibited the *deviatoric stress tensor* d_{ij}, which contains the viscous stress. We first show that d_{ij}, and hence σ_{ij}, must be a *symmetric* tensor. We can do that by considering figure 6.2. We show a square parcel of fluid of side Δ. We show those forces on each face which exert a torque about the z-axis. We see that the torque is $\Delta^3(\sigma_{21} - \sigma_{12})$, since each face has area Δ^2 and each of the four forces considered has a moment of $\Delta/2$ about the z-axis. Now this torque must be balanced by the angular acceleration of the parcel about the z-axis. Now the moment of inertial of the parcel is a multiple of Δ^4. As $\Delta \to 0$ we see that the angular acceleration must tend to infinity as Δ^{-1}. It follows that the only way to have stability of a parcel is for $\sigma_{21} = \sigma_{12}$. The same argument applies to moments about the other axes.

A final requirement we shall place on d_{ij}, so a further condition on the fluids we shall study, is that there should be no preferred direction, the condition of *isotropy*. The conditions of isotropy of symmetric matrices of second order then imply that d_{ij} can satisfy these while being linear in the components of the velocity derivative matrix only if it has either of two forms:

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad \frac{\partial u_k}{\partial x_k}\delta_{ij}. \tag{6.3}$$

For a Newtonian fluid the linearity implies that the most general allowable deviatoric stress has the form

$$d_{ij} = \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\Big) + \mu'\frac{\partial u_k}{\partial x_k}\delta_{ij}. \tag{6.4}$$

Notice that we have divided the two terms so that the first term, proportional to μ, has zero trace. Thus if $\mu' = 0$, the deviatoric stress contributes nothing to the normal force on an area element; this is given solely by the pressure force. The possibility of a normal force distinct from the pressure force is allowed by the second term of (6.4). We have attached the term viscosity to μ, so μ' is usually called the *second viscosity*. Often it is taken as zero, an approximation that is generally valid for liquids. The condition $\mu' = 0$ is equivalent to what is sometimes called the *Stokes relation*. In gases in particular μ' may be positive, in which case the thermodynamic pressure and the normal stresses are distinct.

It should be noted that if we had simply taken d_{ij} to be proportional to the velocity derivative matrix, then the splitting (3.5) would show that only e_{ij} could possibly appear, since otherwise uniform rotation of the fluid would produce a force orthogonal to the rotation axis, , which is never observed. The second term in (6.4) then follows as the only isotropic symmetric tensor linear in the velocity derivative which could be included as a contribution to "pressure".

In this course we shall be dealing with two special cases of (6.4). The first is an incompressible fluid, in which case

$$\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad \text{(incompressible fluid)}. \tag{6.5}$$

Note that with the incompressibility

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i. \tag{6.6}$$

The second case is compressible flow in one space dimension. Then $\mathbf{u} = (u(x,t), 0, 0)$ and the only non-zero component of the stress tensor is

$$\sigma_{11} = -p + \mu''\frac{\partial u}{\partial x}, \quad \mu'' = \frac{4}{3}\mu + \mu', \quad \text{one} - \text{dimensional gas flow}. \tag{6.7}$$

The momentum balance equation in the form

$$\rho\frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{6.8}$$

together with the stress tensor given by (6.4), defines the momentum equation for the *Navier-Stokes* equations. These are the most commonly used equations for the modeling of the rheology of fluids. The have been found to apply to a wide variety of practical problems, but it is important to realize their limitations. First, for highly rarified gases the mean free path of molecules of the gas can become so large that the concept of a fluid parcel, small with respect to the macroscopic scale but large with respect to mean free path, becomes untenable. Also, many common fluids, honey being an example, are non-Newtonian and can exhibit effects not captured by the Navier-Stokes equations. Finally, whenever a flow involves very small domains of transition, the Navier-Stokes model may break down. Example of this occurs in shock waves in gases, where changes occur over a distance of only several mean free paths, and in the interface between fluids, which can involve transitions over distances comparable to intermolecular scales. In these problems a multi-scale analysis is usually needed, which can bridge the macroscopic-molecular divide.

Finally, we point out that the viscosities in this model will generally depend upon temperature, but for simplicity we shall neglect this variation, and in particular for the incompressible case we always take μ to be constant. Also we shall often exhibit the *kinematic viscosity* $\nu = \frac{\mu}{\rho}$ in place of μ. We remark that ν has dimensions length^2/ time, as can be verified from the momentum equations after division by ρ.

6.2 Some examples of incompressible viscous flow

We now take the density and viscosity to be constant and consider several exact solutions of the incompressible Navier-Stokes equations. We shall be dealing with fixed or moving rigid boundaries and we need the following assumption regarding the boundary condition on the velocity in the Navier-Stokes model:

Assumption (The non-slip condition): At a rigid boundary the relative motion of fluid and boundary will vanish.

Thus at a non-moving rigid wall the velocity of the fluid will be zero, while at any point on a moving boundary the fluid velocity must equal the velocity of that point of the boundary. This condition is valid for gases and fluids in situations where the stress tensor is well approximated by (6.4). It can fail in small domains and in rarified gases, where some slip may occur.

6.2.1 Couette flow

Imagine two rigid planes $y = 0, H$ where the no-slip condition will be applied. The plane $y = H$ moves in the x-direction with constant velocity U, while the plane $y = 0$ is stationary. The flow is steady, so the velocity field must be a function of y alone. Assuming constant density, $\mathbf{u} = (u(y), 0)$ and $p_x = 0$ we obtain a momentum balance if

$$-\mu u_{yy} = 0. \tag{6.9}$$

Thus given that $u(0) = 0, u(H) = U$, we have $u = Uy/H$. We see that the viscous stress is here constant and equal to $\mu U/H$. This is the force per unit area felt by the plane $y = 0$. No pressure gradient is needed to sustain this stress field. Couette flow is the simplest exact solution of the Navier-Stokes equations with non-zero viscous stress.

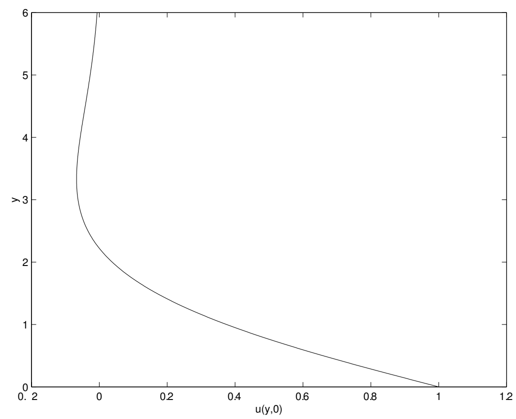

Figure 6.3: The velocity in the Rayleigh problem at t=0 mod 2π, y in units of $\sqrt{\mu/\omega}$.

6.2.2 The Rayleigh problem

A related unsteady problem results from the time dependent motion in the x-direction with velocity $U(t)$of the plane $y = 0$. A no-slip condition is applied on this plane. A fluid of constant density occupies the semi-infinite domain $y > 0$. In this case an exact solution of the Navier-Stokes equations is provided by $\mathbf{u} = (u(y, t), 0), p = 0$, with

$$u_t - \mu u_{yy} = 0,\ u(0) = U(t). \tag{6.10}$$

In the case $U(t) = U_0 \cos \omega t$ we see that $u(y, t) = \Re(e^{i\omega t} f(y))$ where $f(y)$ is the complex-valued function of y satisfying

$$i\omega f - \mu f_{yy} = 0,\ f(0) = U_0. \tag{6.11}$$

We shall also require that $u(\infty) = 0$. Thus

$$u = \Re U_0 e^{i\omega t-(1+i)y\sqrt{\frac{\omega}{2\nu}}} = U_0 \cos\big(\omega t - \sqrt{\frac{\omega}{2\nu}} y\big) e^{-y\sqrt{\frac{\omega}{2\nu}}}. \tag{6.12}$$

We show the velocity field in figure 6.3. Note that the oscillation dies away extremely rapidly, with barely one reversal before decay is almost complete.

6.2.3 Poiseuille flow

We consider now a flow in a cylindrical geometry. A Newtonian viscous fluid of constant density is in steady motion down a cylindrical tube of radius R and of infinite extent in both directions. Because of viscous stresses at the walls of the tube, we expect there to be a pressure gradient down the tube. Let the axis of the tube be the z-axis, r the radial variable, and $\mathbf{u} = (u_z, u_r, u_\theta) = (u_z(r), 0, 0)$ the velocity field in cylindrical polar coordinates. We note here, for future reference, the form of the Navier-Stokes equations in these coordinates:

$$\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu \nabla^2 u_z, \tag{6.13}$$

$$\frac{\partial u_r}{\partial t} + \mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = \nu\Big(Lu_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\Big), \tag{6.14}$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u} \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} + \frac{1}{r\rho}\frac{\partial p}{\partial \theta} = \nu\Big(Lu_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\Big), \tag{6.15}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0. \tag{6.16}$$

Here

$$u \cdot \nabla = u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}, \tag{6.17}$$

$$\nabla^2 = \frac{\partial^2(\cdot)}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial(\cdot)}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2(\cdot)}{\partial \theta^2}, \quad L = \nabla^2 - \frac{1}{r^2}. \tag{6.18}$$

For the problem at hand, we set $p = -Gz+$ constant to obtain the following equation for $u_z(r)$:

$$\mu \nabla^2 u_z = -G = \mu\Big(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\Big). \tag{6.19}$$

The no-slip condition applies at $r = R$, so the relevant solution of (6.19) is

$$u_z = \frac{G}{4\mu}(R^2 - r^2). \tag{6.20}$$

Thus the velocity profile is parabolic. The total flux down the tube is

$$Q \equiv 2\pi \int_0^R r u_z dr = \frac{\pi G R^4}{8\mu}. \tag{6.21}$$

If a tube of length L is subjected to a pressure difference Δp at the two ends, then we can expect to drive a total volume flow or flux $Q = \frac{\pi \Delta p R^4}{8\mu L}$ down the tube. The rate W at which work is done to force the fluid down a tube of length L is the pressure difference between the ends of the tube times the volume flow rate Q, i.e.

$$W = \frac{\pi G^2 L R^4}{8\mu} \tag{6.22}$$

Poiseuille flow can be easily observed in the laboratory, particularly in tubes of small radius, and measurements of flow rates through small tubes provides one way of determining a fluid's viscosity. Of course all tubes are finite, the velocity profile (6.20) is not established at once when fluid is introduced into a tube. This *entry effect* can persist for substantial distances down the tube, depending on the viscosity and the tube radius, and also on the velocity profile at the entrance. Another interesting question concerns the *stability* of Poiseuille flow in a doubly infinite pipe; this was studied by the engineer Osborne Reynolds in the 1870's. He observed instability and transition to turbulence in long tubes. An application of Poiseuille flow of some importance is to blood flow; and in the arterial system there are many branches which are too short to escape significant entry effects.

A generalization of Poiseuille flow to an arbitrary cylinder, bounded by generators parallel to the z-axis and having a cross section S is easily obtained. The equation for u_z is now

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = -G/\mu, \quad u_z = 0 \text{ on } \partial S. \tag{6.23}$$

The solution is necessarily ≥ 0 for $G > 0$ and can be found by standard methods for the inhomogeneous Laplace equation.

6.2.4 Flow down an incline

We consider now the flow of a viscous fluid down an incline, see figure 6.4. The velocity has the form $(u, v, w) = (u(z), 0, 0)$ and the pressure is a function of z alone. The fluid is forced down the incline by the gravitational body force. The equations to be satisfied are

$$\rho g \sin\alpha + \mu \frac{d^2 u}{dz^2} = 0, \quad \frac{dp}{dz} + \rho g \cos\alpha = 0. \tag{6.24}$$

On the free surface $z = H$ the stress must equal the normal stress due to the constant pressure, p_0 say, above the fluid. Thus $\sigma_{xz} = \nu \frac{du}{dz} = 0$ and $\sigma_{zz} = -p = -p_0$ when $z = H$. Since the no-slip condition applies, we have $u(0) = 0$. Therefore

$$u = \frac{\rho g \sin\alpha}{2\mu} z(2H - z), \quad p = p_0 + \rho g (H - z)\cos\alpha. \tag{6.25}$$

The volume flow per unit length in the y-direction is

$$\int_0^U u dz = \frac{g H^3 \sin\alpha}{3\nu}. \tag{6.26}$$

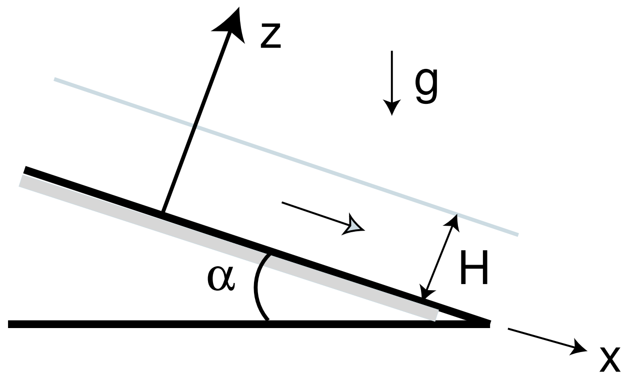

Figure 6.4: Flow of a viscous fluid down an incline.

6.2.5 Flow with circular streamlines

We consider a velocity field in cylindrical polar coordinates of the form $(u_z, u_r, u_\theta) = (0, 0, u_\theta(r,t))$, with $p = p(r,t)$. From (6.13)-(6.18) the equation for u_θ is

$$\frac{\partial u_\theta}{\partial t} = \nu\Big(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\Big), \tag{6.27}$$

with the equation

$$\frac{\partial r}{\partial r} = \frac{\rho}{r} u_\theta^2 \tag{6.28}$$

determining the pressure. The vorticity is

$$\omega = \frac{1}{r}\frac{\partial r u_\theta}{\partial r}. \tag{6.29}$$

From (6.27) we then find an equation for the vorticity

$$\frac{\partial \omega}{\partial t} = \nu\Big(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\Big) = \nu \nabla^2 \omega. \tag{6.30}$$

This equation, which is the symmetric form of the heat equation in two space dimensions, may be used to study the decay of a point vortex in two dimensions, see problem 6.2.

6.2.6 The Burgers vortex

Th implication of (6.30) is that vorticity confined to circular streamlines in two dimensions will diffuse like heat, never reaching a non-trivial steady state in R^2. We now consider a solution of the Navier-Stokes equations which involves a two-dimensional vorticity field $\omega = (\omega_z, \omega_r, \omega_\theta)) = (\omega(r), 0, 0)$. The idea is to prevent the vorticity from diffusing by placing it in a steady irrotational flow field of the form $(u_z, u_r, u_\theta) = (\alpha z, -\alpha r/2, 0)$. Thus the full velocity field has the form

$$(u_z, u_r, u_\theta) = (\alpha z, -\alpha r/2, u_\theta(r,t)). \tag{6.31}$$

Now the z-component of the vortiity equation is, with (6.31),

$$\frac{\partial \omega}{\partial t} - \frac{\alpha}{2}\frac{r}{}\frac{\partial \omega}{\partial r} - \alpha\omega = \nu \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \omega}{\partial r}\Big), \quad \omega = \frac{1}{r}\frac{\partial r u_\theta}{\partial r}. \tag{6.32}$$

First note that if $\nu = 0$, so that there is no diffusion of ω, we my solve the equation to obtain

$$\omega = e^{\alpha t} F(r^2 e^{\alpha t}), \tag{6.33}$$

where $F(r^2)$ is the initial value of ω. This solution exhibits the exponential growth of vorticity coming from the stretching of vortex tubes in the straining flow $(\alpha z, -\alpha r/2, 0)$.

If now we restore the viscosity, we look for a *steady* solution of (6.32), representing a vortex in for which diffusion is balanced by the advection of vorticity toward the z-axis. We have

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{\alpha}{2}r^2\omega + \nu r\frac{\partial \omega}{\partial r}\Big) = 0. \tag{6.34}$$

Integrating and enforcing the condition2 that $r^2\omega$ and $r\frac{\partial \omega}{\partial r}$ vanish when $r = \infty$, we have

$$\frac{\alpha}{2} r\omega + \nu\frac{d\omega}{dr} = 0. \tag{6.35}$$

Thus

$$\omega(r) = Ce^{\frac{-\alpha r^2}{4\nu}}, \tag{6.36}$$

so that

$$u_\theta = \frac{\Gamma}{2\pi}\frac{1 - e^{\frac{-\alpha r^2}{4\nu}}}{r}, \tag{6.37}$$

where we have redefined the constant to exhibit the total circulation of the vortex. Note that as ν decreases the size of the vortex tubes shrinks. With Γ fixed this would mean that the vorticity of the tube is increased.

6.2.7 Stagnation-point flow

In this example we attempt to modify the two-dimensional stagnation point flow with streamfunction $UL^{-1}xy$ to a solution in $y > 0$ of the Navier-Stokes equations with constant density, satisfying the no-slip condition on $y = 0$. The vorticity will satisfy

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} - \nu\Big(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\Big) = 0. \tag{6.38}$$

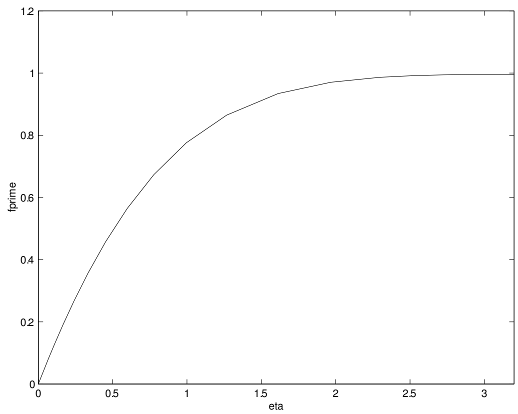

Figure 6.5: f' versus η for the viscous stagnatioin point flow.

If we set $\psi = UL^{-1}xF(y)$, then $\omega = -UL^{-1}yF''$. Insertion in (6.38) gives

$$F'F'' - FF''' - Re^{-1}F'''', \tag{6.39}$$

where $Re = UL/\nu$. The boundary conditions are that $F(0) = F'(0) = 0$ to make ψ, u, v vanish on the wall $y = 0$, and $F \sim y$ as $y \to \infty$, so that we obtain the irrotational stagnation point flow at $y = \infty$.

One integration of (6.39) can be carried out to obtain

$$F'^2 - FF'' - Re^{-1}F''' = 1. \tag{6.40}$$

With $F = Re^{-1/2}f(\eta), \eta = Re^{1/2}y$, (6.40) becomes

$$f'^2 - ff'' - f''' = 1, \tag{6.41}$$

with conditions $f'(\infty) = 1, f(0) = f'(0) = 0$. We show in figure 6.5 the solution $f'(\eta)$ of this ODE problem. This represents a gradual transition through a layer of thickness of order $\sqrt{UL/nu}$ between the null velocity on the boundary and the velocity $U(x/L)$ which u has at the wall in the irrotational stagnation point flow. We shall be returning to a discussion of such transition layers in chapter 7, where we take up the study of boundary layers.

6.3 Dynamical similarity

In the stagnation point example just considered, the dimensional combination $Re = UL/\nu$ has occurred as a parameter. This parameter, called the *Reynolds* *number* in honor of Osborne Reynolds, arose because we chose to exhibit the problem in a dimensionless notation. Consider now the Navier-Stokes equations with constant density it their dimensional form:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{6.42}$$

We may define dimensionless (starred) variables as follows:

$$\mathbf{u}^* = \mathbf{u}/U, \mathbf{x}^* = \mathbf{x}/L, p^* = p/\rho U^2. \tag{6.43}$$

Here U, L are assumed to be a velocity and length characteristic of the problem being studied. In the case of flow past a body, L might be a body diameter and U the flow speed at infinity. In these starred variables it is easily checked that the equations become

$$\frac{\partial \mathbf{u}^*}{\partial t} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + \nabla^* p^* - \frac{1}{Re} \nabla^{*2} \mathbf{u}^* = 0, \quad \nabla^* \cdot \mathbf{u}^* = 0. \tag{6.44}$$

Thus Re survives as the only dimensionless parameter in the equations. For a given value of Re a given problem will have a solution or solutions which are fully determined by the value of Re.[3] Nevertheless the set of solutions is fully determined by Re and Re alone. Thus we are able to make a correspondence between various problems having different U and L but the same value of Re. We call this correspondence *dynamical self-similarity.* Two flows which are self-similar in this respect become identical which expressed in the starred, dimensionless variables (6.43). In a sense the statement "the viscosity ν is small" conveys no dynamical information, although the intended implication might be that $Re \gg 1$. If L is also "small", then it could well be that $Re = 1$ or $e \ll 1$. The only meaningful way to state that a fluid is "almost inviscid" is through the Reynolds number, $Re \gg 1$. If we want to consider fluids whose viscosity is dominant compared to inertial forces, we should require $Re \ll 1$. These remarks underline the oft-repeated definition of Re as "the ratio of inertial to viscous forces". This is because

$$\frac{\rho \mathbf{u} \cdot \nabla \mathbf{u}}{\mu \nabla^2 \mathbf{u}} = Re \frac{\mathbf{u}^* \cdot \nabla^* \mathbf{u}^*}{\nabla^{*2} \mathbf{u}^*} \sim Re \tag{6.45}$$

since we regard all starred variables as of order unity.

example 6.1: The drag D per unit length of a circular cylinder of radius L in a two-dimensional uniform flow of speed U must satisfy $D = \rho U^2 L F(Re)$ for some function F. Note that we are assuming here that cylinders are fully determined by their radius. In experiments other factors, such as surface material or roughness, slight ellipticity, etc. must be considered.

Problem set 6

[3]It is not always the case that well-formulated boundary-value problems for the Navier-Stokes equations have unique solutions. See the example of viscous flow in a diverging channel, page 79 of Landau an Lifshitz.

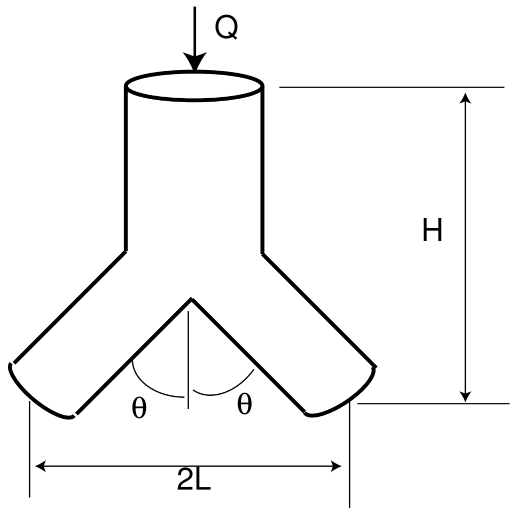

Figure 6.6: Bifurcating Poiseuill flow. Assume a parabolic profile in each section.

1. Consider the following optimzation problem: A Newtonian viscous fluid of constant density flows through a cylindrical tube of radius R_1, which then bifurcates into two straight tubes of radius R_2, see the figure. A volume flow Q is introduced into the upper tube, which divides into flows of equal flux $Q/2$ at the bifurcation. Because of the material composition of the tubes, it is desirable that the wall stress $\mu du/dr$, evaluated at the wall, be the same in both tubes. If L and H are given and fixed, what is the angle θ which minimizes the rate of working required to sustain the flow Q?. Be sure to verify that you have a true minimum.

2. Look for a solution of (6.30) of the form $\omega = t^{-1}F(r/\sqrt{t})$, satisfying $\omega(\infty,t) = 0$, $2\pi\int_0^\infty r\omega(r,t)dr = 1, t >)$. Show, by computing u_θ with $u_\theta(\infty,t) = 0$, that this represents the decay of a point vortex of unit strength in a vbiscous fluid, i.e.

$$\lim_{t\to 0+} u_\theta(r,t) = \frac{1}{2\pi r}, r > 0. \tag{6.46}$$

3. A Navier-Stokes fluid has constant ρ, μ, no body forces. Consider a motion in a fixed bounded domain V with no-slip condition on its rigid boundary. Show that

$$dE/dt = -\Phi, E = \int_V \rho|\mathbf{u}|^2/2dV, \Phi = \mu\int_V (\nabla\times\mathbf{u})^2 dV.$$

This shows that for such a fluid kinetic energy is converted into heat at a rate $\Phi(t)$. This last function of time gives the net *viscous dissipation* for the fluid contained in V. (Hint: $\nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u}$. Also $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \nabla\times\mathbf{A}\cdot\mathbf{B} - \nabla\times\mathbf{B}\cdot\mathbf{A}$.)

4. In two dimensions, with streamfunction ψ, where $(u, v) = (\psi_y, -\psi_x)$, show that the incompressible Navier-Stokes equations without body forces for a fluid of constant ρ, μ reduce to

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{(\partial(\psi, \nabla^2\psi)}{\partial(x, y)} - \nu\nabla^4\psi = 0.$$

In terms of ψ, what are the boundary conditions on a rigid boundary if the no-slip condition is satisfied there?

5. Find the time-periodic 2D flow in a channel $-H < y < H$, filled with viscous incompressible fluid, given that the pressure gradient is $dp/dx = A + B\cos(\omega t)$, where A, B, ω are constants. This is an oscillating 2D Poiseuille flow. You may assume that $u(y, t)$ is even in y and periodic in t with period $2\pi/\omega$.

6. verify (6.33).

7. The plane $z = 0$ is rotating about the z-axis with an angular velocity Ω. A Newtonian viscous fluid of constant density and viscosity occupies $z > 0$ and the fluid satisfies the no-slip condition on the plane, i.e. at $z = 0$ the fluid rotates with the plane. By centrifugal effect we expect the fluid near the plane to be thrown out radially and a compensating flow of fluid downward toward the plane.

Using cylindrical polar coordinates, look for a steady solution of the Navier-Stokes equations of the form

$$(u_z, u_r, u_\theta) = (f(z), rg(z), rh(z)). \tag{6.47}$$

We assume that the velocity component u_θ vanishes as $z \to \infty$. Show that then

$$\frac{p}{\rho} = \nu\frac{df}{dz} - \frac{1}{2}f^2 + F, \tag{6.48}$$

where F is a function of r alone. Now argue that, if $h(\infty) = 0$, i.e. no rotation at infinity, then F must in fact be a constant. From the r and θ component of the momentum equation together with $\nabla \cdot \mathbf{u} = 0$, find equations for f, g, h and justify the following conditions:

$$f = \frac{df}{dz} = 0, h = \Omega, \quad z = 0; \quad f', h \to 0, \quad z \to \infty. \tag{6.49}$$

(The solution of these equations is discussed on pp. 75-76 of L&L and 290-92 of Batchelor.)