Chapter 4

Potential flow

Potential or irrotational flow theory is a cornerstone of fluid dynamics, for two reasons. Historically, its importance grew from the developments made possible by the theory of harmonic functions, and the many fluids problems thus made accessible within the theory. But a second, more important point is that potential flow is actually realized in nature, or at least approximated, in many situations of practical importance. Water waves provide an example. Here fluid initially at rest is set in motion by the passage of a wave. Kelvin's theorem insures that the resulting flow will be irrotational whenever the viscous stresses are negligible. We shall see in a later chapter that viscous stresses cannot in general be neglected near rigid boundaries. But often potential flow theory applies away from boundaries, as in effects on distant points of the rapid movements of a body through a fluid.

An example of potential flow in a barotropic fluid is provided by the theory of sound. There the potential is not harmonic, but the irrotational property is acquired by the smallness of the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in the momentum equation. The latter thus reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho} \nabla p \approx 0. \tag{4.1}$$

Since sound produces very small changes of density, here we may take ρ to be will approximated by the constant ambient density. Thus $\mathbf{u} = \nabla \phi$ with $\frac{\partial \phi}{\partial t} = -p/\rho$.

4.1 Harmonic flows

In a potential flow we have

$$\mathbf{u} = \nabla \phi. \tag{4.2}$$

We also have the Bernoulli relation (for body force $\mathbf{f} = -\rho \nabla \Phi$)

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + \int \frac{dp}{\rho} + \Phi = 0. \tag{4.3}$$

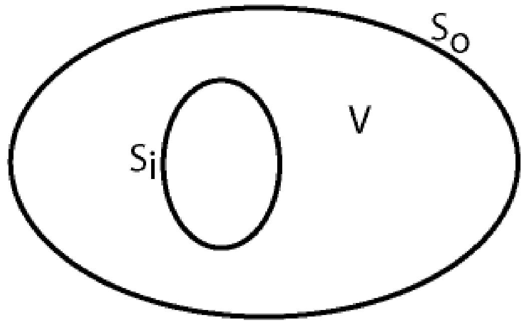

Figure 4.1: A domain V, bounded by surfaces $S_{i,o}$ where $\frac{\partial\phi}{\partial n}$ is prescribed.

Finally, we have conservation of mass

$$\rho_t + \nabla \cdot (\rho \nabla \phi) = 0. \tag{4.4}$$

The most extensive use of potential flow theory is to the case of constant density, where $\nabla \cdot \mathbf{u} = \nabla^2 \phi = 0$. These *harmonic flows* can thus make use of the highly developed mathematical theory of harmonic functions. in the problems we study here we shall usually consider explicit examples where existence is not an issue. On the other hand the question of uniqueness of harmonic flows is an important issue we discuss now. A typical problem is shown in figure 4.1.

A harmonic function ϕ has prescribed normal derivatives on inner and outer boundaries S_i, S_o of an annular region V. The difference $\mathbf{u}_d = \nabla \phi_d$ of two solutions of this problem will have zero normal derivatives on these boundaries. That the difference must in fact be zero throughout V can be established by noting that

$$\nabla \cdot (\phi_d \nabla \phi_d) = (\nabla \phi_d)^2 + \phi_d \nabla^2 \phi_d = (\nabla \phi_d)^2. \tag{4.5}$$

The left-hand side of (4.5) integrates to zero over V to zero by Gauss' theorem and the homogeneous boundary conditions of $\frac{\partial \phi_d}{\partial n}$. Thus $\int_V (\nabla \phi_d)^2 dV = 0$, implying $\mathbf{u}_d = 0$.

Implicit in this proof is the assumption that ϕ_d is a single-value function. A function ϕ is single-valued in V if and only if $\oint_C d\phi = 0$ on any closed contour C contained in V. In three dimensions this is insured by the fact that any such contour may be shrunk to a point in V. In two dimensions, the same conclusion applies to *simply-connected* domains. In non-simply connect domains uniqueness of harmonic flows in 2DS is not assured. Note for a harmonic flow

$$\oint_C d\phi = \oint_C \mathbf{u} \cdot d\mathbf{x} = \Gamma_C, \tag{4.6}$$

so that a potential which is not single valued is associated with a non-zero circulation on some contour. Since there is no vorticity within the domain of harmonicity, we must look outside of this domain to find the vorticity giving rise to the circulation.

Example 4.1: The point vortex of problem 1.2 is an example of a flow harmonic in a non-simply connected domain which excludes the origin. If $\mathbf{u} = \frac{1}{2\pi}(-y/r^2, x/r^2)$ then the potential is $\frac{\theta}{2\pi} + constant$ and the circulation on an simply closed contour oriented counter-clockwise is 1. This defines the *point vortex of unit circulation*. Here the vorticity is concentrated at the origin, outside the domain of harmonicity.

Example 4.2 Steady two-dimensional flow harmonic flow with velocity $(U, 0)$ at infinity, past a circular cylinder of radius a centered at the origin, is not unique. The flow of example 2.4 plus an arbitrary multiple of the point vortex flowof example 4.1 will again yield a flow with the same velocity at infinity, and still tangent to the boundary $r = a$:

$$\phi = Ux(1 + a^2/r^2) + \frac{\Gamma}{2\pi}\theta. \tag{4.7}$$

4.1.1 Two dimensions: complex variables

In two dimensions harmonic flows can be studied with the powerful apparatus of complex variable theory. We define the *complex potential* as an analytic function of the complex variable $z = x + iy$:

$$w(z) = \phi(x, y) + i\psi(x, y). \tag{4.8}$$

We will suppress t in our formulas in the case when the flow is unsteady. If we identify ϕ with the potential of a harmonic flow, and ψ with the stream function of the flow, then by our definitions of these quantities

$$(u, v) = (\phi_x, \phi_y) = (\psi_y, -\psi_x), \tag{4.9}$$

yielding the Cauchy-Riemann equations $\phi_x = \psi_y, \phi_y = -\psi_x$. The derivative of w gives the velocity components in the form

$$\frac{dw}{dz} = w'(z) = u(x, y) - iv(x, y). \tag{4.10}$$

Notice that the Cauchy-Riemann equations imply that $\nabla\phi \cdot \nabla\psi = 0$ at every point where the partials are defined, implying that the streamlines are there orthogonal to the lines of constant potential ϕ.

Example 4.3: The uniform flow at an angle α to the horizontal, with velocity $Q(\cos\alpha, \sin\alpha)$ is given by the complex potential $w = Qze^{-i\alpha}$.

Example 4.4: In complex notation the harmonic flow of example 4.2 may be written

$$w = U(z + a^2/z) + \frac{i\Gamma}{2\pi}\log z \tag{4.11}$$

where e.g. we take the principle branch of the logarithm function.

As a result of the identification of the complex potential with an analytic function of a complex variable, the conformal map becomes a valuable tool in

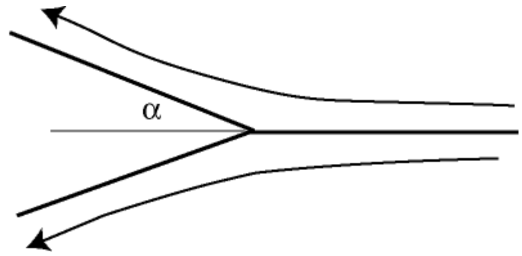

Figure 4.2: Flow onto a wedge of half-angle α.

the construction of potential flows. For this application we may start with the physical of z-plane, where the complex potential $w(z)$ is desired. A conformal map $z \to Z$ transforms boundaries and boundary conditions and leads to a problem which can be solved to obtain a complex potential $W(Z)$. Under the map values of ψ are preserved, so that streamlines map onto streamlines.

Example 4.5: The flow onto a wedge-shaped body (see figure 4.2). Consider in the Z plane the complex potential of a uniform flow,$-UZ, U > 0$. The region above upper surface of the wedge to the left, and the and the positive x-axis to the right, is mapped onto the upper half-plane $Y >$) by the function $Z = z^{\frac{\pi}{\pi-\alpha}}$. Thus $w(z) = -Uz^{\frac{\pi}{\pi-\alpha}}$.

Example 4.6: The map $z(Z) = Z + \frac{b^2}{Z}$ maps the circle of radius $a > b$ in the Z-plane onto the ellipse of semi-major axis $\frac{a^2+b^2}{a}$ and semi-minor (y)-axis $\frac{a^2-b^2}{a}$ in the z-plane. And the exterior is mapped onto the exterior. Uniform flow with velocity $(U, 0)$at infinity, past the circular cylinder $|Z| = a$, has complex potential $W(Z) = U(Z + a^2/Z)$. Inverting the map and requiring that $Z \approx z$ for large $|z|$ gives $Z = \frac{1}{2}(z + \sqrt{z^2 - 4b^2})$. Then $w(z) = W(Z(z))$ is the complex potential for uniform flow past the ellipse. Notice how the map satisfies $\frac{dz}{dZ} \to 1$ as $z \to \infty$ This insures that that infinity maps by the identity and so the uniform flow imposed on the circular cylinder is also imposed on the ellipse.

4.1.2 The circle theorem

We now state a result which gives the mathematical realization of the physical act of "placing a rigid body in an ideal fluid flow", at least in the two-dimensional case.

Theorem 3 *Let a harmonic flow have complex potential $f(z)$, analytic in the domain $|z| \leq a$. If a circular cylinder of radius a is place at the origin, then the new complex potential is $w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$.*

To show this we need to establish that the analytical properties of the new flow match those of the old, in particular that the analytic properties and the singularities in the flow are unchanged. Then we need to verify that the surface of the circle is a streamline. Taking the latter issue first, note that on the circle

$\frac{a^2}{\bar{z}} = z$, so that there we have $w = f(z) + \overline{f(z)}$, implying $\psi = 0$ and so the circle is a streamline. Next, we note that the added term is an analytic function of z if it is not singular at z. (If $f(z)$ is analytic at z, so is $\overline{f(\bar{z})}$. As for the location of singularities of w, since f is analytic in $|z| \leq a$ it follows that $f\left(\frac{a^2}{z}\right)$ is analytic in $|z| \geq a$, and the same is true of $\overline{f\left(\frac{a^2}{\bar{z}}\right)}$. Thus the only singularities of $w(z)$ in $|z| > a$ are those of $f(z)$.

Example 4.7: If a cylinder of radius a is placed in a uniform flow, then $f = Uz$ and $w = Uz + U\overline{(a^2/\bar{z})} = U(z + a^2/z)$ as we already know. If a cylinder is placed in the flow of a point source at $b > a$ on the x-axis, then $f(z) = \frac{Q}{2\pi} \ln(z - b)$ and

$$w(z) = \frac{Q}{2\pi}(\ln(z-b) + \overline{\ln\left(\frac{a^2}{\bar{z}} - b\right)} = \frac{Q}{2\pi}(\ln(z-b) + \ln(z - a^2/b) - \ln z) + C, \quad (4.12)$$

where C is a constant. From this form it may be verified that the imaginary part of w is constant when $z = ae^{i\theta}$. Note that the *image system* of the source, with singularities within the circle, consists of a source of strength Q at the image point a^2/b, and a source of strength $-Q$ at the origin.

Example 4.8: A point vortex at position z_k of circulation Γ_k has the complex potential $w_k(z) = -i\frac{\Gamma_k}{2\pi} \ln(z - z_k)$. A collection of N such vortices will have the potential $w(z) = \sum_{k=1}^{N} w_k(z)$. Since vorticity is a material scalar in two-dimensional ideal flow, and the delta-function concentration may be regarded as the limit of a small circular patch of constant vorticity, we expect that each vortex must move with he harmonic flow created at the vortex by the other $N - 1$ vortices. Thus the positions $z_k(t)$ of the vortices under this law of motion is governed by the system of N equations,

$$\overline{\frac{dz_j}{dt}} = \frac{-i}{2\pi} \sum_{k=1, k\neq j}^{N} \frac{\Gamma_k}{z - z_k}. \quad (4.13)$$

Note the conjugation on the left coming from the identity $w' = u - iv$.

4.1.3 The theorem of Blasius

An important calculation in fluid dynamics is the force exerted by the fluid on a rigid body. In two dimensions and in a steady harmonic flow this calculation can be carried out elegantly using the complex potential.

Theorem 4 *Let a steady uniform flow past a fixed two-dimensional body with bounding contour C be a harmonic flow with velocity potential $w(z)$. Then, if no external body forces are present, the force (X, Y) exerted by the fluid on the body is given by*

$$X - iY = \frac{i\rho}{2} \oint_C \left(\frac{dw}{dz}\right)^2 dz. \quad (4.14)$$

Here the integral is taken round the contour in the counter-clockwise sense. This formula, due to Blasius, reduces the force calculation to a complex contour integral. Since the flow is harmonic, the path of integration may be distorted to any simple closed contour encircling he body, enabling the method of residues to be applied. The exact technique will depend upon whether are not the are singularities in the flow exterior to the body.

To prove the result, first recall that $dX - idY = p(-dy - idx) = -ipd\bar{z}$. Also, Bernoulli's theorem for steady ideal flow applies, so that

$$p = -\frac{\rho}{2}\left|\frac{dw}{dz}\right|^2 + C, \tag{4.15}$$

where clearly the constant C will play no role. Thus

$$X - iY = \frac{i\rho}{2}\oint_C \frac{dw}{dz}\overline{\frac{dw}{dz}}d\bar{z}. \tag{4.16}$$

However, the contour C is a streamline, so that $d\psi = 0$ there, and so on C we have $\overline{\frac{dw}{dz}}d\bar{z} = d\bar{w} = dw = \frac{dw}{dz}dz$. using this in (4.16) we obtain (4.14).

Example 4.9: We have found in problem 2.1 that the force on a circular cylinder in a uniform flow is zero. To verify this using Blasius' theorem, we set $w = U\Big(z + \frac{a^2}{z}\Big)$ so that $U^2\Big(1 - \frac{a^2}{z^2}\Big)2$ is to be integrated around C. Since there is no term proprotional to z^{-1} in the Laurent expansion aboutn the origin, the residue is zero and we get no contribution to the force integral.

Example 4.10: Consider a source of strength Q placed at $(b, 0)$ and introduce a circular cylinder of radius $a < b$ into the flow. From example 4.6 we have

$$\frac{dw}{dz} = \frac{1}{z-b} + \frac{1}{z - a^2/b} - \frac{1}{z}. \tag{4.17}$$

Squaring, we get

$$\frac{1}{(z-b)^2} + \frac{1}{(z-a^2/b)^2} + \frac{1}{z^2} + \frac{2}{(z-b)(z-a^2/b)} - \frac{2}{z(z-a^2/b)} - \frac{2}{z(z-b)}. \tag{4.18}$$

The first three terms to not contibute to the integral around the circle $|z| = a$. For the last three, the partial fraction decomposition is

$$\frac{A}{z-b} + \frac{B}{z-a^2/b} + \frac{C}{z}, \tag{4.19}$$

where we compute $A = \frac{2a^2}{(b^2-a^2)b}, B = \frac{2b^3}{a^2(a^2-b^2)}, C = \frac{2(a^2+b^2)}{a^2b}$. The contributions come from the poles within the circle and we have

$$X - iY = \frac{i\rho}{2}\frac{Q^2}{4\pi^2}2\pi i(B + C) = \frac{Q^2\rho}{2\pi}\frac{a^2}{b(b^2-a^2)}. \tag{4.20}$$

The cylinder is therefore feels a force of attraction toward the source.

This introduction to the use of complex variables in the analysis of two-dimenisonal harmonic flows will provide the groundwork for a discussion of lift and airfoil design, to be taken up in chapter 5.

4.2 Flows in three dimensions

We live in three dimensions, not two, and the "flow past body" problem in two dimensions introduces a domain which is not simply connected, with important consequences. The relation between two and three-dimensional flows is particularly significant in the generation of lift, as we shall see in chapter 5. In the present section we treat topics in three dimensions which are direct extensions of the two-dimensional results just given. They pertain to bodies, such as a sphere, which move in an irrotational, harmonic flow.

4.2.1 The simple source

The source of strength Q in three dimensions satisfies

$$\text{div } \mathbf{u} = Q\delta(\mathbf{x}), \quad \mathbf{u} = \nabla\phi. \tag{4.21}$$

Here $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional delta function. It has the following properties: (i) It vanishes if $\mathbf{x} \neq 0$. (ii) Any integral of $\delta(\mathbf{x})$ over an open region containing the origin yields unity. It is best to think of all relations involving delta functions and other distributions as limits of relations using smooth functions.

In our case, integrating $\nabla^2\phi = Q\delta(\mathbf{x})$ over a sphere of radius $R_0 > 0$ we get

$$\int_{R=R_0} \frac{\partial\phi}{\partial n}\, dS = Q. \tag{4.22}$$

Since $\nabla^2\phi = 0, \mathbf{x} \neq 0$, and since the delta function must be regarded as an isotropic distribution, having no exceptional direction, we make the guess (using now $\nabla^2\phi = R^{-1}d^2(R\phi)/dR^2$) that $\phi = C/R, R^2 = x^2 + y^2 + z^2$ for some constant C. Then (4.22) shows that $C = -\frac{Q}{4\pi}$. Thus the simple source in three dimensions, of strength Q, has the potential

$$\phi = -\frac{Q}{4\pi}\frac{1}{R}. \tag{4.23}$$

Note that Q is equal to the volume of fluid per unit time crossing any deformation of a spherical surface, assuming the deformed surface surounds the origin. [1]

[1] We indicate how to justify this calculation using a limit operation. Define the three-dimensional delta function by $\lim_{\epsilon\to 0}\delta_\epsilon(R)$ where $\delta_\epsilon = \frac{3}{2\pi\epsilon^3}\frac{1}{1+(R/\epsilon)^3}$. Solving $\nabla^2\phi_\epsilon = \delta_\epsilon = R^{-2}\frac{d}{dR}\Big(R^2\frac{d\phi_\epsilon}{dR}\Big)$, under the condition that ϕ_ϵ vanish at infinity, we obtain $\phi_\epsilon = -\frac{1}{4\pi R} + \int_R^\infty R^{-2}\Big[tan^{-1}(R^3\epsilon^{-3}) - \pi/2\Big]dR$. For any positive R the integral tends to zero as $\epsilon \to 0$.

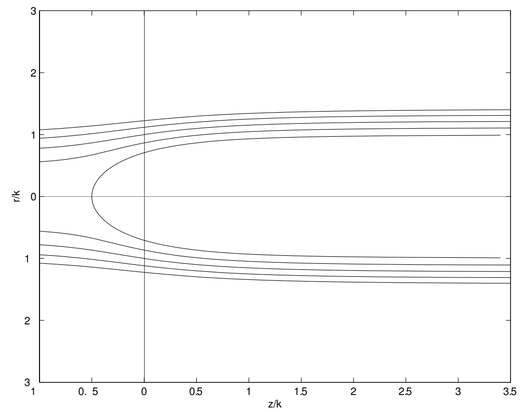

Figure 4.3: The Rankine fairing. All lengths are in units of k.

4.2.2 The Rankine fairing

We consider now a simple source of strength Q placed at the origin in a uniform flow $W\mathbf{i}_z$. The combined potential is then

$$\phi = Uz - \frac{Q}{4\pi}\frac{1}{R}. \tag{4.24}$$

The flow is clearly symmetric about the z-axis. In cylindrical polar coordinates $(z, r, \theta), r^2 = x^2 + y^2$ we introduce again the Stokes stream function ψ:

$$u_z = \phi_z = \frac{1}{r}\frac{\partial \psi}{\partial r},\ u_r = \phi_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{4.25}$$

Thus for (4.24) we have

$$\frac{1}{r}\frac{\partial \psi}{\partial r} = U + \frac{Q}{4\pi}\frac{z}{R^3}. \tag{4.26}$$

Integrating,

$$\psi = Ur^2/2 - \frac{Q}{4\pi}\Big(\frac{z}{R} + 1\Big). \tag{4.27}$$

In (4.27) we have chosen the constant of integration to make $\psi = 0$ on the negative z-axis.

We show the stream surface $\psi = 0$, as well as several stream surfaces $\psi > 0$, in figure 4.3. This gives a good example of a uniform flow over a semi-infinite body. An interesting question is whether or not such a body would experience a force. We will find below that D'Alembert's paradox applies to *finite* bodies in three dimensions, that the drag force is zero, but it is not obvious that the result applies to bodies which are not finite.

We will use this question to illustrate the use of conservation of momentum to calculate force on a distant contour. In figure 4.4 the large sphere S of radius R_0 is centered at the origin and intersects the fairing on the at a circle bounding

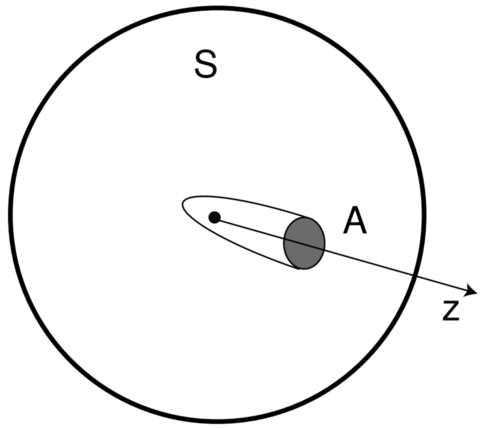

Figure 4.4: Geometry of the momentum integral for computation of the force on the Rankine fairing.

the disc A. Let S' be the spherical surface S minus hat part within the boundary of A. We are considering steady harmonic flow and so the momentum equation may be written

$$\frac{\partial}{\partial x_j}[\rho u_i\, u_j + p\,] = 0. \tag{4.28}$$

Let V' be the region bounded by S' and the piece of fairing enclosed. Integrating (4.28) over V' and using the divergence theorem., the contribution from the surface of the fairing is the integral $-\mathbf{n}p$ over this surface, where $\mathbf{n}$ is the outer normal *of the fairing*. Thus this contribution is the force $\mathbf{F}$ experienced by the enclosed piece of fairing, a force clearly directed along the z axis and therefore equal to the drag, $\mathbf{F} = D\mathbf{i}_z$. The remainder of the integral, taking only the z-component, takes the form of an integral over S minus the contribution from A. Thus conservation of momentum gives

$$D + \rho \int_S u_z \mathbf{u}\cdot\mathbf{R}/R + \frac{1}{2}\big[U^2 - |\mathbf{u}|^2\big]\frac{z}{R} dS - I_A = 0. \tag{4.29}$$

We have here using the Bernoulli formula for the flow, $p + \frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}U^2$, the pressure at infinity being taken to be zero. Treating first the integral over S, we have

$$\mathbf{u} = U\mathbf{i}_z + \frac{Q}{4\pi}\frac{\mathbf{R}}{R^3}, \quad |\mathbf{u}|^2 = U^2 + \frac{UQ}{2\pi}\frac{z}{R^3} + \frac{Q^2}{16\pi^2}\frac{1}{R^4}. \tag{4.30}$$

Thus the integral in question becomes

$$\int_S \Big(U + \frac{Q}{4\pi}\frac{z}{R^3}\Big)\Big(\frac{Uz}{R} + \frac{Q}{4\pi}\frac{1}{R^2}\Big) - \frac{1}{4\pi}\Big(UQ\frac{z^2}{R^4} + \frac{1}{8\pi}\frac{Q^2 z}{R^5}\Big) dS. \tag{4.31}$$

We see that this last integral gives $UQ + \frac{1}{2}UQ - \frac{1}{2}UQ = UQ$. For the contribution I_A, we take the limit $R_0 \to \infty$ to obtain $I_A = U^2\pi r_\infty^2$, where r_∞ is the

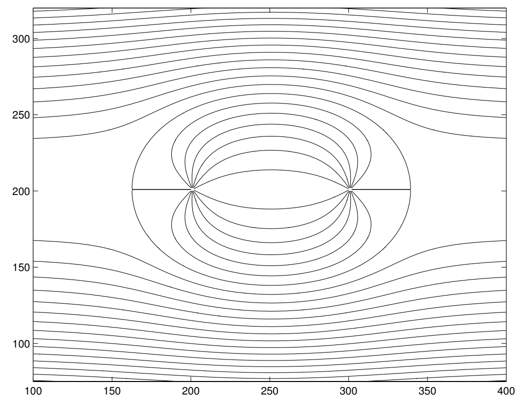

Figure 4.5: Flow around an airship.

asymptotic radius of the airing as $z \to \infty$. In this limit $D' \to D$, the total drag of the fairing. Thus the momentum integral method gives

$$D + UQ - U^2 \pi r_\infty^2 = 0. \tag{4.32}$$

But from (4.27) we see that the stream surface $\psi = 0$ is given by

$$z = \frac{r^2 - \frac{1}{2}k^2}{\sqrt{k^2 - r^2}}, \quad k^2 = \frac{Q}{\pi U}. \tag{4.33}$$

Thus $r_\infty = k$, and (4.32) becomes

$$D + UQ - UQ = D = 0, \tag{4.34}$$

so the drag of the fairing is zero.

Example 4.11: The flow considered now typifies the early attempts to model the pressure distribution of an airship. The model consists of a source of strength Q at position $z = 0$ on the z-axis, and a equalizing sink (source of strength $-Q$) at the pint $z = 1$ on the z-axis. Since the source strengths cancel, a finite body is so defined when the singularities are place in the uniform flow $U\mathbf{i}_z$. It can be shown (see problem 4.7 below), that stream surfaces for the flow are given by constant values of

$$\Psi = \frac{U}{2} R^2 \sin^2\theta - \frac{Q}{4\pi}\Big(\cos\theta + \frac{1 - R\cos\theta}{\sqrt{R^2 - 2R\cos\theta + 1}}\Big), \tag{4.35}$$

where R, θ are spherical polars at the origin, with axial symmetry. We show the stream surfaces in figure 4.4.

4.2.3 The Butler sphere theorem.

The circle theorem for two-dimensional harmonic flows has a direct analog in three dimensions.

Theorem 5 *Consider an axisymmetric harmonic flow in spherical polars* (R, θ, φ)*,* $u_\varphi = 0$*, with Stokes stream function* $\Psi(R, \theta)$ *vanishing at the origin:*

$$u_R = \frac{1}{R^2 \sin\theta} \frac{\partial \Psi}{\partial \theta}, u_\theta = \frac{-1}{R \sin\theta} \frac{\partial \Psi}{\partial R}. \tag{4.36}$$

If a rigid sphere of radius a *is introduced into the flow at the origin, and if the singularities of* Ψ *exceed* a *in distance from the origin, then the stream function of the resulting flow is*

$$\Psi_s = \Psi(R, \theta) - \frac{R}{a} \Psi(a^2/R, \theta). \tag{4.37}$$

It is clear that Ψ_s vanishes when $R = a$, so the surface of the sphere is a stream surface. Also the added term introduces no new singularities outside the sphere. Thus the theorrm is proved if we can verify that $\frac{R}{a}\Psi(a^2/R, \theta)$ represents a harmonic flow. In spherical polars with axial symmetry the only component of vorticity is

$$\omega_\varphi = \frac{1}{R}\Big[\frac{\partial(R u_\theta)}{\partial R} - \frac{\partial u_R}{\partial \theta}\Big]. \tag{4.38}$$

Thus the condition on Ψ for an irrotational flow is

$$R^2 \frac{\partial^2 \Psi}{\partial R^2} + \sin\theta \frac{\partial}{\partial \theta}\Big(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\Big) \equiv L_R \Psi = 0. \tag{4.39}$$

If $\frac{R}{a}\Psi(a^2/R, \theta)$ is inserted into (4.39) we can show that the equation is satisfied provided it is satisfied by $\Psi(R, \theta)$, see problem 4.8. Finally, since $\Psi(R, \theta)$ vanishes at the origin at least as R, $R\Psi(a^2/R, \theta$ is bounded at infinity and velocity component must decay as $O(R^{-2})$, so the uniform flow there is undisturbed.

Example 4.12: A sphere in a uniform flow $U\mathbf{i}_z$ has Stokes stream function

$$\Psi(R, \theta) = \frac{U}{2} R^2 \sin^2\theta \Big[1 - \frac{a^3}{R^3}\Big]. \tag{4.40}$$

This translates into the following potential:

$$\phi = Uz\Big(1 + \frac{1}{2}\frac{a^3}{R^3}\Big). \tag{4.41}$$

Example 4.13: Consider a source of strength Q place on the z axis at $z = b$ and place a rigid sphere of radius $a < b$ at the origin. The streamfunction for this source which vanishes at the origin is

$$\Psi(R, \theta) = -\frac{Q}{4\pi}(\cos\theta_1 + 1), \tag{4.42}$$

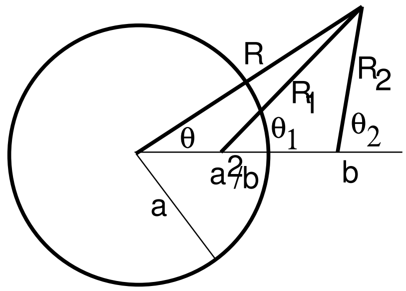

Figure 4.6: A a sphere of radius a in the presence of a source at $z = b > a$.

where θ_1 is defined in figure 4.6.

Now from the law of cosines and figure 4.6 we have

$$R\cos\theta - b = R_1\cos\theta_1, \quad R_1^2, R_1^2 = b^2 - 2bR\cos\theta + R^2. \tag{4.43}$$

Thus

$$\cos\cos\theta_1 = \frac{R\cos\theta - b}{\sqrt{b^2 - 2bR\cos\theta + R^2}}. \tag{4.44}$$

Thus the stream function including the sphere, Ψ_s, is given by

$$\Psi_s = -\frac{Q}{4\pi}\Big[\frac{R\cos\theta - b}{\sqrt{b^2 - 2bR\cos\theta + R^2}} + 1\Big] + \frac{Q}{4\pi}\frac{R}{a}\Big[\frac{\frac{a^2}{R}\cos\theta - b}{\sqrt{b^2 - 2b\frac{a^2}{R}\cos\theta + \frac{a^4}{R^2}}} + 1\Big]. \tag{4.45}$$

Now, again using the law of cosines, $\sqrt{b^2 - 2b\frac{a^2}{R}\cos\theta + \frac{a^4}{R^2}} = bR_2/R$. Also we may use $R^2 = R_2^2 + 2\frac{a^2}{b}R\cos\theta - \frac{a^4}{b^2}$. Then Ψ_s may be brought into the form

$$\Psi_s = -\frac{Q}{4\pi}\Big[\frac{R\cos\theta - b}{R_1} + 1\Big] - \frac{a}{b}\frac{Q}{4\pi}\Big[\frac{R\cos\theta - \frac{a^2}{b}}{R_2}\Big] + \frac{Q}{4\pi}\Big[\frac{R - R_2}{a}\Big]. \tag{4.46}$$

The first term on the right is the source of strength Q at $z = b$. The second term is another source, of strength $\frac{a}{b}Q$, at the image point $z = a^2/b$. The last term can be understood as a line distribution of sinks of density $\frac{Q}{4\pi a}$, extending from the origin to the image point a^2/b. Indeed, if a point P on this line segment is associated with an angle θ_P, the the contribution from such a line of sinks would be

$$\frac{Q}{4\pi a}\int_0^{\frac{a^2}{b}} \cos\theta_P dz. \tag{4.47}$$

But $dR = -\cos\theta_P dz$, so the integral becomes

$$-\frac{Q}{4\pi a}\int_R^{R_2} dR = \frac{Q}{4\pi a}(R - R_2). \tag{4.48}$$

4.3 Apparent mass and the dynamics of a solid body in a fluid

Although harmonic flow is an idealization never realized exactly in actual fluids (except in some cases of super fluid dynamics), it is a good approximation in many fluid problems, particularly when rapid changes occur. A good example is the abrupt movement of a solid body through a fluid, for example a swimming stroke of the hand. We know from experience that a abrupt movement of the hand through water gives rise to a force opposing the movement. It is easy to see why this must be, within the theory of harmonic flows. An abrupt movement of the hand through still water causes the fluid to move relative to a observer fixed with the still fluid at infinity. This observer would therefore compute at the instant the hand is moving a finite kinetic energy of the fluid, whereas before the movement began the kinetic energy was zero. To produce this kinetic energy work must have been done, and so a force with a finite component opposite to the direction of motion must have occurred. We are here dealing only with the fluid, but if the body has mass the clearly a force is also needed to accelerate that mass. Thus both the body mass and the fluid movement contribute to the force experienced.

In a harmonic flow we shall show that, in the absence of external body forces, the force on a rigid body is proportional to its acceleration, and further the force contributed by the fluid can be expressed as an addition, *apparent* mass of the body. In other words the augmented force due to the presence of the surrounding fluid and the energy it acquires during motion of a body, can be explained as an inertial force associated with additional mass and the work done against that force. The term *virtual mass* is also used to denote this apparent mass. For a sphere, which has an isotropic geometry with no preferred direction, the apparent mass is just a scalar to be added to the physical mass. In general, however, the apparent mass associated with translation of a body in two or three dimensions will depend on the direction of the velocity vector. It thus must be a second order tensor, represented by the *apparent mass matrix.*

4.3.1 The kinetic energy of a moving body

Consider an ideal fluid at rest and introduce a moving rigid body, in two or three dimensions. An observer at rest relative to the fluid at infinity will se a disturbance of the flow which vanishes at infinity. It would be natural to compute compute the momentum of this motion by calculating the integral $\int \rho \mathbf{u} dV$ of the region exterior to the body. The problem is that such harmonic flows have an expansion at infinity of the form

$$\phi \sim a \ln r - \mathbf{A} \cdot \mathbf{r} r^{-2} + O(r^{-2}) \tag{4.49}$$

in two dimensions and

$$\phi \sim \frac{a}{R} - \mathbf{A} \cdot \mathbf{R} R^{-3} + O(R^{-3}) \tag{4.50}$$

in three dimensions. Thus

$$\rho \int \nabla \phi dV = \int_S \phi \mathbf{n} dS, \tag{4.51}$$

where S comprises both a surface in a neighborhood of infinity as well as the body surface, is not absolutely convergent as the distant surfaces recedes. We point out that $a = 0$ in two dimensions if the area of the body is fixed and there is no circulation about the body. In three dimensions a vanishes if the body has fixed volume, see problem 4.12.

But even if $a = 0$ and $\phi = O(R^{-1})$ the value of the integral is only conditionally convergent will depend on how one defines the distant surface. So the value attributed to the fluid momentum is ambiguous by this calculation.

An unambiguous result is however possible, if we instead focus on the kinetic energy and from it determine the incremental momentum created by a change in velocity. Let us fix the orientation of the body and consider its movement through space, without rotation. This *translation* is completely determined by a velocity vector $\mathbf{U}(t)$. The, from the discussion of section 2.6 we know that a harmonic flow will satisfy the instantaneous boundary condition

$$\frac{\partial \phi}{\partial n} = \mathbf{U}(t) \cdot \mathbf{n} \tag{4.52}$$

on the surface of the body. Now $\nabla^2 \phi = 0$ is a linear equation, and so we see that there must exist a vthe Φ_i as encoding the effect of the shape of the body from all possible harmonic flows associated with translation of the body.

We may now compute the kinetic energy E of the fluid exterior to the body using

$$\mathbf{u} = U_i \nabla \Phi_i. \tag{4.53}$$

Thus

$$E(t) = \frac{1}{2} M_{ij} U_i U_j, \quad M_{ij} = \rho \int \nabla \Phi_i \cdot \nabla \Phi_j dV. \tag{4.54}$$

the integral being over the fluid domain. Clearly the matrix M_{ij} is symmetric, and thus

$$dE = M_{ij} U_j dU_i. \tag{4.55}$$

On the other hand the change of kinetic energy, dE, must equal, in the absence of external body forces, the work done by the force $\mathbf{F}$ which the body exerts on the fluid, $dE = \mathbf{F} \cdot \mathbf{U} dt$. But according to Newton's second law, the incremental momentum $d\mathbf{P}$ is given by $d\mathbf{P} = \mathbf{F} dt$. Consequently $dE = \mathbf{U} \cdot d\mathbf{P}$. From (4.54) we thus have

$$dE = \rho M_{ij} dU_j U_i = dP_i U_i. \tag{4.56}$$

Since this holds for arbitrary $\mathbf{U}$ we must have $dP_i = M_{ij} dU_j$. Integrating and using the fact that M_{ij} is independent of time and $\mathbf{P} = 0$ when $\mathbf{U} = 0$ we obtain

$$P_i = M_{ij} U_j. \tag{4.57}$$

Thus we have reduced the problem of computing momentum, and then the inertial force, to calculating M_{ij}. Since M_{ij} arises here as an effective mass term associated with movement of the body, it is called the *apparent mass matrix.*

But the calculation of M_{ij} is not ambiguous since the integral for the kinetic energy converges absolutely, and we can deduce M_{ij} once the energy is written in the form (4.54). We write

$$E = \frac{\rho}{2}\int_V |\mathbf{u}|^2 dV = \frac{\rho}{2}\int_V (\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}+\mathbf{U})dV + \frac{\rho}{2}\int_V |\mathbf{U}|^2 dV. \qquad (4.58)$$

The reason for this splitting is to exhibit $\mathbf{u}-\mathbf{U}$, whose normal component will vanish on the body by (4.52). Now $\mathbf{u}+\mathbf{U} = \nabla(\phi+\mathbf{U}\cdot\mathbf{x}$ and $\mathbf{u}-\mathbf{U}$ is solenoidal, so $\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}+\mathbf{U}\nabla\cdot\big[(\phi+\mathbf{U}\cdot\mathbf{x})(\mathbf{u}-\mathbf{U})\big]$. Thus, remembering that $|\mathbf{U}|^2$ is a constant, the application of the divergence theorem and use of (4.52) on the inner boundary aallows us to reduce (4.58) to

$$E = \frac{\rho}{2}\int_{S_o} (\phi+\mathbf{U}\cdot\mathbf{x})(\mathbf{u}-\mathbf{U})\cdot\mathbf{n}dS + |\mathbf{U}|^2(\mathcal{V}-\mathcal{V}_b), \qquad (4.59)$$

where S_0 is the outer boundary, $\mathcal{V}$is the volume contained by S_o, and $\mathcal{V}_0$ is the volume of the body.

To compute the integral in (4.59) we need only the leading term of ϕ. Referring to (4.49),(4.50), we note that $a = 0$ for a finite rigid body (or even for a flexible body of constant area/volume), see problem 4.11. Using

$$\phi = -\frac{\mathbf{A}\cdot\mathbf{x}}{|\mathbf{x}|^N}, \quad \mathbf{u} = \frac{-\mathbf{A}}{|\mathbf{x}|^N} + \frac{N\mathbf{A}\cdot\mathbf{x}\,\mathbf{x}}{|\mathbf{x}|^{N+2}} \qquad (4.60)$$

in (4.59) we have

$$E \sim \frac{\rho}{2}\int_{S_o}\Big[\frac{-\mathbf{A}\cdot\mathbf{x}}{|\mathbf{x}|^N} + \mathbf{U}\cdot\mathbf{x}\Big]\Big[\frac{-\mathbf{A}}{|\mathbf{x}|^N} + \frac{N\mathbf{A}\cdot\mathbf{x}\,\mathbf{x}}{|\mathbf{x}|^{N+2}} - \mathbf{U}\Big]\cdot\mathbf{n}dS. \qquad (4.61)$$

We are free to choose S_0 to be a sphere of radius R_o. The term quadratic in $\mathbf{A}$ in (4.61) is $O(R_o^{1-2N})$and so the contribution is of order R_o^{-N} and will vanish in the limit. The term under the integral quadratic in $\mathbf{U}$ yields $-|\mathbf{U}|^2\mathcal{V}$, thus canceling part of the last term in (4.59). Finally two of the cross terms in $\mathbf{U},\mathbf{A}$ cancel out, the remaining term giving rthe contribution $2\pi\rho(N-1)\mathbf{A}\cdot\mathbf{U}$. Thus

$$E = \frac{\rho}{2}\big[2\pi\rho(N-1)\mathbf{A}\cdot\mathbf{U} - \mathcal{V}_b|\mathbf{U}|^2\big]. \qquad (4.62)$$

Since $\phi = \Phi_i U_i$, we may write $A_i(t) = \rho^{-1}m_{ij}U_j$ where m_{ij} is dependent on body shape but not time. Then

$$E = \frac{1}{2}\big[2\pi(N-1)m_{ij} - \mathcal{V}_b\rho\delta_{ij}\big]U_iU_j. \qquad (4.63)$$

Comparing (4.63) and (4.54) we obtain an expression for the apparent mass matrix:

$$M_{ij} = 2\pi(N-1)m_{ij} - \mathcal{V}_b\rho\delta_{ij}, \quad N = 2, 3. \qquad (4.64)$$

We thus can obtain the apparent mass of a body by a knowledge of the expansion of ϕ in a neighborhood of infinity.

Given that we have computed a finite fluid momentum we are in a position to state

Theorem 6 *(D'Alembert's paradox) In a steady flow of a perfect fluid in three dimensions, and in steady flow in two dimensions for a body with zero circulation, the force experienced by the body is zero.*

Clearly if the flow is steady $d\mathbf{P}/dt = \mathbf{F} = 0$, and we are done. Of course the proof hinges on the existence of a finite fluid momentum associated with a single-value potential function.

Example 4.14: To find the apparent mass matrix of a elliptic cylinder in two dimensions, we may use example 4.6. In the Z-plane the complex potential for uniform flow $-Q(\cos\theta, \sin\theta)$ past the cylinder of radius $a > b$ is $W(Z) = -Qe^{-i\theta}Z - Qe^{i\theta}s^a/Z$. Since $Z = \frac{1}{2}(z + \sqrt{z^2 - 4b^2})$ we may expand at infinity to get

$$w(z) \sim -Qe^{-i\theta}z - Q\big[\frac{a^2e^{i\theta} - b^2e^{-i\theta}}{z}\big] + \dots, \tag{4.65}$$

so that

$$\mathbf{A} = \big[U(a^2 - b^2), V(a^2 + b^2)\big], \;\; (U, V) = Q(\cos\theta, \sin\theta). \tag{4.66}$$

Now the ellipse intersects the positive x-axis at its semi-major axis $\alpha = \frac{a^2+b^2}{a}$, and the positive y-axis at its semi-minor axis $\beta = \frac{a^2-b^2}{a}$. From (4.66) we obtain the apparent mass matrix

$$\mathbf{M} = 2\pi\rho\begin{pmatrix} a^2 - b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} - \pi\frac{a^4 - b^4}{a^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pi\rho\begin{pmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}. \tag{4.67}$$

In particular for a circular cylinder the apparent mass is just the mass of of the fluid displaced by the body.

An alternative expression for the apparent mass matrix in terms of an integral over the surface of the body rather than a distant surface is readily obtained in terms of the potential Φ_i. We have

$$E = \frac{\rho}{2}\int_V \nabla\Phi_i \cdot \nabla\Phi_j U_i U_j dV = \frac{\rho}{2}\int_V \nabla \cdot \Phi_j \nabla\Phi_i] dV U_i U_j. \tag{4.68}$$

Applying the divergence theorem to the integral, surfaces S_o, s_B, and observing that $\Phi_i \nabla\Phi_j = O(|\mathbf{x}|^{1-2N})$, we see that the receding surface integral will give zero contribution. Recalling that $\frac{\partial\phi}{\partial n} = \mathbf{U}\cdot\mathbf{n}$ on the body surface, we see that $\frac{\partial\Phi_i}{\partial n} = n_i$ *where the normal is directed out of the body surface.* In applying the divergence theorem the normal at the body is into the body, with the result that (4.54) applies with

$$M_{ij} = -\rho\int_{S_b} \Phi_j n_i dS, \;\; \mathbf{n} \text{ directed out of the body.} \tag{4.69}$$

It follows from (4.57) that the fluid momentum is given by

$$\mathbf{P} = -\rho \int_{S_b} \phi \mathbf{n} dS. \tag{4.70}$$

We can verify the fact that (4.70) gives the fluid momentum by taking its time derivative, using the result of problem 1.6:

$$\frac{d}{dt} \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} \frac{\partial \phi}{\partial t} \mathbf{n} dS + \int_{S_b} (\mathbf{u} \cdot \mathbf{n}) \nabla \phi dS. \tag{4.71}$$

Using the Bernoulli theorem for harmonic flow we have

$$\frac{d}{dt} \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} \Big[-\frac{p}{\rho} - \frac{1}{2}|\mathbf{u}|^2 \Big] \mathbf{n} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{u} dS. \tag{4.72}$$

Converting the terms on the right involving $\mathbf{u}$ to a volume integral, we observe that the latter converges absolutely at infinity, as so we have, for the integration over the domain exterior to S_b,

$$\int_V \big[\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{2} \nabla |\mathbf{u}|^2 \big] dV = -\int_V \mathbf{u} \times (\nabla \times \mathbf{u}) dV = 0. \tag{4.73}$$

Therefore

$$-\frac{d}{dt} \rho \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} p \mathbf{n} dS = \mathbf{F}, \tag{4.74}$$

where $\mathbf{F}$ is the force applied by the body to the fluid.

Finally we note again that the inertial force required to accelerate a body in a perfect fluid will contain a contribution from the actual mass of the body, M_b. This mass appears as an additional term $M_b \delta_{ij}$ in the expression (4.64) for the apparent mass matrix. The total momentum of the body including its apparent mass is thus $I_i = M_{ij} U_j + M_b U_i$ and Newton's second law becomes

$$\frac{dI_i}{dt} = F_i, \tag{4.75}$$

where $\mathbf{F}$ is the force applied to the body, to accelerate it and the surrounding fluid.

4.3.2 Moment

We have so far restricted the motion of the body to translation, i.e. with no rotation relative to the fluid at infinity. In general a moment is experienced by a body in translational motion, so that in fact a free body will rotate and thereby give the apparent mass matrix a dependence upon time. The theory may be easily extended to include a time dependent apparent mass, due either to rotation and/or deformation of he body, see section 4.4. But even in steady translational motion of a body, a non-zero moment can result, see problem 4.14. (There is no D'Alembert paradox for moment.)

For example, in analogy with (4.69), the *apparent angular momentum* of the fluid exterior to a body is defined by

$$\mathbf{P}_A = -\rho \int_{S_b} \phi(\mathbf{x} \times \mathbf{n}) dS, \tag{4.76}$$

the normal being out of the body. It may be shown in a manner similar to that used for linear momentum that

$$\frac{d\mathbf{P}_A}{dt} = \mathbf{T}, \tag{4.77}$$

where $\mathbf{T}$ is the torque applied to the fluid by the body.

4.4 Deformable bodies and locomotion in an ideal fluid

It might be thought that, in an ideal, or more suggestively, a "slippery" fluid, it would be impossible for a body to locomote, i.e. to "swim" by using some kind of mechanism involving changes of shape. The fact is, however, that inertial forces alone can allow a certain kind of locomotion. The key point is that the flow remains irrotational everywhere, and this will have the effect of disallowing the possibility of the body producing an average force on the fluid which can then accelerate the body. Rather, it is possible to locomote in the sense of getting from point A to point B, put without any finite average acceleration. If the body is assumed to deform periodically over some cycle of configurations, then the kind of locomotion we envision is of a finite, periodic translation (and possible rotation) of the body, repeated with each cycle of deformation.

We first note that the Newtonian relationships that we derived above for a rigid body carry over to an arbitrary deformable body, which for simplicity we take to have a fixed area/volume. This follows immediately from our verification of $\frac{d\mathbf{P}}{dt} = \mathbf{F}$ from (4.70), since we made no assumption about the velocity of the body surface.

Now the idea behind inertial swimming is to deform the body in a periodic cycle which causes a net translation. To simplify the problem we consider only a simple traslation of a suitable symmetric body along a line, e.g a body symmetric about the z-axis, translating with velocity $U(t)$along this axis. In general we cannot expect the velocity to remain of one sign, but over on cycle there will be a positive translation, say to the right. Let $U_m(t)$ be the velocity of the center of mass of the body, and let $U_v(t)$ be the velocity of the center of volume of the body. Also let P_D be the momentum of deformation of the body *relative to its center of volume*. If the total mass of the body is m, then $U_m(t)m$ is the momentum of the body mass. Consider now the momentum of the fluid. If the apparent mass of the body (now a scalar $M(t)$) is multiplied by $U_v(t)$, we get the fluid momentum associated with the instantaneous motion of the shape of the body at time t. Finally, we have the momentum associated with

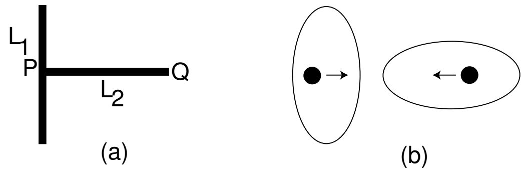

Figure 4.7: Swimming in an ideal fluid.

the motion of the boundary of the body relative to the center of volume. If the potential of this harmonic flow of deformation is ϕ_D, then the deformation momentum is $P_D(t) = -\rho \int_{S_b} \phi_D \mathbf{n} \cdot \mathbf{i} dS$. The total momentum if body and fluid is thus $mU_m(t) + M(t)U_v(t) + P_D$. If initially the fluid and body is at rest, then this momentum, which is conserved, must vanish, and it is for this reason that locomotion is possible.

Consider first a body of uniform density. so the center of mass and of volume coincide. The $U_m = U_v = U$ and

$$U(t) = \frac{P_D(t)}{m + M(t)}. \tag{4.78}$$

There is no reason for the right-hand side o (4.78) to have non-zero time average, and when it does not, we call this *locomotion by squirming*. To see squirming in action it is best to treat an simple case, see example 4.15 below.

Alternatively, we can imagine that the center of mass changes relative to the center of volume without and deformation. Then deformation occurs giving a new shape, then the center of mass again changes relative to the center of volume holding the boy fixed in the new shape. If the two shapes lead to different apparent masses, locomotion occurs by *recoil swimming*, see example 4.16.

Example 4.15: We show in figure 4.7(a) a squirming body of a simplified kind. The body consists of a then vertical strip of length $L_1(t)$, and a horizontal part of length $L_2(t)$. The length will change as a function of time, think of L_2 as being extruded from the material of L_1. We neglect the width w of the strips except when computing mass and volume. The later are constant, implying $L_1 + L_2 = L$ is constant. The density of the material is taken as ρ_b, so the total mass is $M_b = \rho_b wL$ and the total volume is wL.

A cycle begins with $L_1 = L$, when L_2 begins to grow to the right. If $X(t)$ denotes the position of the point P, then $(\rho_b wL_1 + \rho L_1^2/4)\frac{dX}{dt}$ is the momentum of the fluid and vertical segment, where we have used the formula for apparent mass of a flat plate in 2D. The velocity of the extruded strip varies linearly from $\frac{dX}{dt}$ at P to $\frac{d(X+L_2)}{dt}$ at Q, so the momentum of the horizontal part is $\rho_b wL_2 d(X + \frac{1}{2}L_2)/dt$, where we neglect the apparent mass of the extruded strip. The first half of the cycle stops when $L_1 = 0$. Assuming the start is from fluid and body at rest, the sum of these momenta remains zero throughout the

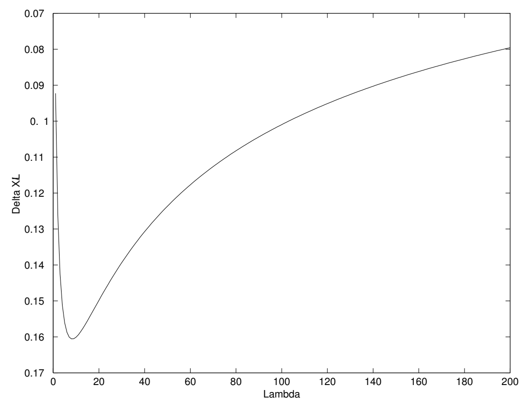

Figure 4.8: $\Delta X/L$ versus λ for the model squirmer of figure 4.7(a).

half-cycle:

$$(\rho_b wL + \rho L_1^2/4)\frac{dX}{dt} + \frac{\rho_b w L_2}{2}\frac{dL_2}{dt} = 0. \tag{4.79}$$

If we et$L_2 = Lt/T, L_1 = L(1-t/T)$ where T is the half-period of the cycle, then we may obtain the change ΔX of X over the half-cycle by quadrature:

$$\Delta(X) = \frac{1}{\lambda}\ln(1+\lambda) - \frac{1}{\sqrt{\lambda}}\tan^{-1}(\sqrt{\lambda}), \quad \lambda = \frac{\rho L}{4\rho_b w}. \tag{4.80}$$

We show this relation in figure 4.8 So we see that at the end of the half-cycle the point P has moved a distance $-\Delta X$ to the left. At this point, we imagine another half-cycle in which L_1 is created at the expense of L_2, but *at the point* Q. Observe that at the start of the second half-cycle Q is located a distance $L + \Delta X$ from the initial position of P. It can be seen from considerations of symmetry that the point Q will move to the left a distance $-\Delta X$ in time T over the second half-cycle. The the cycle is complete, $L_2 = L$, abnd the midpoint can be relabeled P. Thus the net advance to the right of the point P in time $2T$ has been $L - 2\Delta x$, which from figure 4.8 always exceeds about $.68L$.

Example 4.16: Recoil swimming can be illustrated by the 2D model of Figure (4.7)b. Let P denote the center of an elliptical surface of major,minor semi-axes α, β. Within this body is a mass M on a bar enabling it to be driven to the right or left. The weight of the shell and mechanism is m. Let the position of the center be $X(t)$ and the position of the mass be $x(t)$. At the beginning of the half-cycle the mass lies a distance $\beta/2$ to the right of P and the ellipse has its major axis vertical. The mass the moves to the left a distance β. Since momentunm is conserved, we have

$$(m+\rho\pi\alpha^2)\frac{dX}{dt}+M(\frac{d(X+x)}{dt}=0. \tag{4.81}$$

Thus over a half-cycle $(m+\rho\pi\alpha^2)\Delta X+M(\Delta X+\Delta x)=0$ or, since $\delta x=\beta$,

$$\Delta X_1=-\frac{M\beta}{m+M+\rho\pi\alpha^2)}. \tag{4.82}$$

at this point the surface of the body deforms in a symmetric way, the points $(0,\pm\alpha/2$ moving down to $(0,\pm\beta/2$ and the points $(\pm\beta/2,0$ moving our to $(\pm\alpha/2,0)$, so that the major and minor axes get interchanged. There is no movement of P during this process. No the mass is moved back, a distance β to the left. We see that in this second half-cycle the displacement is

$$\Delta X_2=\frac{M\beta}{m+M+\rho\pi\beta^2)}. \tag{4.83}$$

The displacement over one cycle is then

$$\Delta X=\Delta X_1+\Delta X_2=\frac{M\beta}{m+M+\rho\pi\beta^2)}-\frac{M\beta}{m+M+\rho\pi\alpha^2)}, \tag{4.84}$$

which is positive since $\beta<\alpha$.

Problem set 4

1. (a) Show that the complex potential $w=Ue^{i\alpha}z$ determines a uniform flow making an angle α with respect to the x-axis and having speed U.

(b) Describe the flow field whose complex potential is given by

$$w=Uze^{i\alpha}+\frac{Ua^2e^{-i\alpha}}{z}.$$

2. Recall the system (4.13) governing the motion of point vortices in two dimensions. (a) Using these equations, show that two vortices of equal strengths rotate on a circle with center at the midpoint of the line joining them, and find the speed of their motion.

(b) Show that two vortices of strengths γ and $-\gamma$ move together on straight parallel lines perpendicular to the line joining them. Again find the speed of their motion.

3. Using the method of Blasius, show that the moment of a body in 2D potential flow, about the axis perpendicular to the plane (positive counter-clockwise), is given by

$$M=-\frac{1}{2}\rho Re[\int_C z(dw/dz)^2dz],$$

where Re denotes the real part and C is any simple closed curve about the body. Using this, verify by the residue method that the moment on a circular cylinder with a point vortex of circulation Γ at its center, in uniform flow, experiences zero moment.

4. Compute, using the Blasius formula, the force exerted by a point vortex at the point $c = be^{i\theta}, b > a$ upon a circular cylinder at the origin of radius a. The complex potential of a point vortex at c is $\frac{-\Gamma i}{2\pi} \ln(z - c)$. (Use the circle theorem and residues). Verify that the cylinder is pushed away from the vortex.

5. Prove Kelvin's minimum energy theorem: In a simply-connected domain V let $\mathbf{u} = \nabla\phi, \nabla^2\phi = 0$, with $\partial\phi/\partial n = f$ on the boundary S of V. (This $\mathbf{u}$ is unique in a simply-connected domain). If $\mathbf{v}$ is any differentiable vector field satisfying $\nabla \cdot \mathbf{v} = 0$ in V and $\mathbf{v} \cdot \mathbf{n} = f$ on S, then

$$\int_V |\mathbf{v}|^2 dV \geq \int_V |\mathbf{u}|^2 dV.$$

(Hint: Let $\mathbf{v} = \mathbf{u} + \mathbf{w}$, and apply the divergence theorem to the cross term.)

6. Establish (4.33) and work though he details of the proof of zero drag of the Rankine fairing using the momentum integral method, as outlined in section 4.2.2.

7. In spherical polar coordinates (r, θ, φ) a Stokes stream function Ψ may be defined by $u_R = \frac{1}{R^2 \sin\theta} \frac{\partial\Psi}{\partial\theta}, u_\theta = \frac{-1}{R \sin\theta} \frac{\partial\Psi}{\partial R}$.) Show that in spherical polar coordinates, the stream function Ψ for a source of strength Q, placed at the origin, normalized so that $\Psi = 0$ on $\theta = 0$, is given by $\Psi = \frac{Q}{4\pi}(1 - \cos\theta)$. Verify that the stream function in spherical polars for the airship model consisting of equal source and sink of strength Q, the source at the origin and the sink at $R = 1, \theta = 0$, in a uniform stream with stream function $\frac{1}{2}UR^2(\sin\theta)^2$, is given by (4.35). (Suggestion: The sink will involve the angle with respect to $R = 1, \theta = 0$. Use the law of cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$ for a triangle with θ opposite side c) to express Ψ in terms of R, θ.)

8. In the Butler sphere theorem, we needed the following result: Show that $\Psi_1(R, \theta) \equiv \frac{R}{a}\Psi(\frac{a^2}{R}, \theta)$ is the stream function of an irrotational, axisymmetric flow in spherical polar cordinates, provided that $\Psi(R, \theta)$ is such a flow. (Hint: Show that $L_R\Psi_1(R, \theta) = L_{R_1}\Psi(R_1, \theta)$, where $R_1 = a^2/R$. Here L_R is defined by (4.39).)

9. (Reading, Milne-Thomson sec. 13.52 on "stationary vortex filaments in the presence of a circular cylinder" in 3rd edition.) Consider the following model of flow past a circular cylinder of radius a with two eddies downstream of the body. Consider two point vortices, of opposite strengths, the upper vortex having clockwise circulation $-\Gamma$(i.e. $\Gamma > 0$) located at the point $c = be^{i\theta}$, thus adding a term $(i\Gamma/2\pi)\ln(z - c)$ to the complex potential w, the other being having circulation Γ at the point $\bar{c} = be^{-i\theta}$. Here $b > a > 0$.

Using the circle theorem, write down the complex potential for the entire flow field, and determine by differentiation the complex velocity. Sketch the positions of the vortices and all vortex singularities within the cylinder, indicating their strengths.

10. Continuing problem 9, verify that $x = \pm a, y = 0$ remain stagnation points of the flow. Show that the vortices will remain stationary behind the cylinder (i.e. not move with the flow) provided that

$$U(1 - \frac{a^2}{c^2}) = \frac{i\Gamma}{2\pi} \frac{(c^2 - a^2)(b^2 - a^2) + (c - \bar{c})^2 a^2}{(c - \bar{c})(c^2 - a^2)(b^2 - a^2)}.$$

Show (by dividing both sides of the last equation by their conjugates and simplifying the result) that this relation implies $b - a^2/b = 2b \sin\theta$, that is, the distance between the exterior vortices is equal to the distance between a vortex and its image vortex.

11. Show that the apparent mass matrix for a sphere is $M_0/2\delta_{ij}$ where M_0 is the mass of fluid displaced by the sphere.

12. Show that for a body which may have a time-dependent shape but is of fixed area/volume, the quantity a in (4.49),(4.50) must vanish.

13. Using the alternative definition (4.69), show that M_{ij} is a symmetric matrix.

14. Let the elliptic cylinder of examples 4.14 and 5.13 be place in a steady uniform flow (U, V). Show, using the result of problem 4.3, that the moment experienced by the cylinder is $-\pi\rho UV(\alpha^2 - \beta^2)$, α, β being the major and minor semi-axes of the ellipse.