1. Show that, if $\vec{u}_i, i = 1,2,3$ denotes the velocity field given by

$$u_i = [\frac{\partial^2 \chi}{\partial x_i \partial x_j} - \nabla^2 \chi \delta_{ij}] a_j$$

with $\vec{a} = \vec{i}_1, \vec{i}_2, \vec{i}_3$ respectively and the same function χ in each case, then $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + 2\nabla\chi$ is divergence-free and has the form of the poloidal compomponent with $P = \chi$. Similarly, show that, if $\vec{u}_i, i = 1,2,3$ denotes the velocity field $\vec{b} \times \nabla\psi_i$ with $\vec{b} = \vec{i}_1, \vec{i}_2, \vec{i}_3$ respectively and the same function ψ in each case, then $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3$ is divergence-free and has the form of the toroidal component of velocity with $T = \psi$.

2.Using the same basic approach as for the uniqueness proof for the Stokes equations, prove the following result: Let a finite rigid 3D body move with steady velocity $\vec{U}$ in a fluid otherwise at rest, From the steady Navier-Stokes equations for a fluid of constant density, prove that, if D denotes the drag on the body,

$$UD = \frac{\mu}{2} \int_V \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}^2 \Big) dV.$$

Here V is the domain exterior to the body. The quantity on the right is the total viscous dissipation in the fluid, so this is a mechanical energy equation, stating that the work done on the fluid by the body is egual to the rate of heating of the fluid by viscous dissipation. What conditions on the decay at infinity are needed in this proof? Show that, if the body surface is in motion with an arbitrary velocity $u_S(\vec{x})$, the viscous dissipation equals the rate of working of the surface S on the fluid.

3.Using the form of solution given in terms of the Stokes steamfunction Ψ, derive the Stokes drag law $D = 6\pi\mu r_0 U$ by integration of the stresses over the surface of the sphere. Show in the process that the pressure is responsible for 1/3 of the total drag. You will use

$$\sigma_{r\theta} = \mu r \frac{\partial}{\partial r}\Big(\frac{\partial u_\theta}{r}\Big) + \frac{1}{r}\frac{\partial u_r}{\partial \theta},$$

$$\sigma_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r},$$

together with the divergence conditions on $\vec{u}$.

4. Let the motion of a helical flagellum be given by $(U + V)\vec{i} - Q\vec{t}$ plus a rotation $\big(0, -\Omega b \sin k(s + Qt), \Omega b \cos k(s + Qt)\big)$. Show that the total thrust on length L of flagellum thrust T is given by

$$L^{-1}T = U[(K_T - K_N)\alpha^2 + K_N] + (V - \alpha\Omega k^{-1})[(K_T - K_N)(\alpha^2 - 1)].$$

Show also that the torque m_x about the axis (the x-component)is given by

$$\frac{m_x}{b^2 kL} = U\alpha(K_T - K_N) + (V - \alpha\Omega k^{-1})[\alpha(K_T - K_N) - \alpha^{-1}K_T].$$

From these two expressions, show that simultaneous torque and thrust balance, $T = m_x = 0$, is impossible for an isolated flagellum, in that if both expressions vanish we must have$U = V - \alpha\Omega k^{-1} = 0$. Thus swimming is possible only if there is a passive body attached to the flagellum which can resist torque, or else there is surface rotation of the flagellum itself. Also, if gravity is present a *keel*, i.e. an asymmetric weight distribution, can be used to obtain a restoring torque.