Liquidity-adjusted Expected Shortfall

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1 Introduction

In today's post-Lehman finance service industry, risk management has become more important than ever. One of the important risk measurements in financial risk management is Value at Risk (VaR). VaR quantifies the risk of a portfolio by determining the potential loss over a given period and under a certain confidence level. VaR was initially introduced to measure market risk by JP Morgan in 1992(See Orlova 2008). VaR does not, however, incorporate liquidity risk for the portfolio. Liquidity risk can be understood as (i) a widening of the bid/offer spread as the quantities transacted are larger or (ii) a delayed or protracted liquidation period for securities held in the portfolio.

How to incorporate liquidity risk into a risk management model? Some clues come from the work of Almgren and Chriss who, in 2000, proposed an optimal trade execution model which takes into account price impact as well as market risk. The model aims to minimize a combination of price risk and costs due to price impact (Almgren and Chriss 2000). By minimizing a quadratic utility function assuming simple a linear market impact model, they calculated optimal execution strategies for trading stocks.

In this paper we formulate a related optimization problem to quantify simultaneously the market and liquidity risk of financial assets. For this purpose, rather than considering market impact, we assume that, given an initial position, we will trade only a limited amount of stocks each day until liquidation is completed.[1]

To formulate the optimization problem, we consider the problem mitigating the expected shortfall (Conditional Value At Risk) for an agent who is liquidating a portfolio subject to daily bound on the number of shares that he or she can trade. The idea is that the limit may correspond to a fraction of the historical average or median trading volume, and hence is an observable quantity. Also, we do not impose a fixed time horizon for liquidation.

Given a random variable X, the expected shortfall of Conditional Value at Risk is defined as

$$ES_{\alpha}(X) = E(X|X \leq \gamma)$$

[1] For simplicity, we shall consider stocks (equities). The methods described here extend to other securities, including derivatives. See Acerbi and Tasche (2001) .

where

$$\gamma = VaR_\alpha(X) = \inf\{l \in \boldsymbol{R}: P(X \leq l) > 1 - \alpha\}.$$

We will assume that we have a portfolio of N stocks, with price processes

$$\big(P_1(t), P_2(t), \dots, P_N(t)\big), \dots t = 1,2,,3, \dots$$

The variable *t* represents days. The initial endowment in each stock is denoted by $Q_i, i = 1, .., N$, and the endowments at time *t* are

$$\big(Q_1(t), Q_2(t), \dots, Q_N(t)\big), \dots t = 1,2,,3, \dots$$

According to this notation, and neglecting transaction costs, the profit-loss for the liquidation strategy from date 0 to date t is given by

$$R(t) = \sum_{s=1}^{t-1} \sum_{i=1}^{N} Q_i(s)\big(P_i(s) - P_i(s-1)\big).$$

We assume that the liquidation terminates at some terminal time T, but this time may depend on the strategy itself, i.e. we do not assume that there is a constraint on the time horizon.

We also assume that the liquidation strategy has trading limits, i.e. that for each security there exists a constant k_i such that

$$|Q_i(t+1) - Q_i(t)| \leq k_i \,. \qquad (1)$$

Our objective is to solve the optimization problem

$$Maximize \quad ES_\alpha\left(\min_{t \leq T} R(t)\right) \qquad (2)$$

over all strategies $\big(Q_1(t), Q_2(t), \dots, Q_N(t)\big)$, $t \leq T$, *(with the time horizon depending on Q), with* $\big(Q_1(0), Q_2(0), \dots, Q_N(0)\big)$ *given and subject to the constraint (1).*

Remarks:

1. It can be shown that this problem is well-posed and admits typically a unique solution Q^*. The reason is that -$ES_\alpha(\quad)$ is a sub-additive functional, since it is a coherent risk-measure in the sense of Artzner et al. This implies that -$ES_\alpha(\min_{t\leq T} R(t))$ is convex in Q. Thus, generically, there is a unique minimum and the problem is well-posed and continuous as a function of the initial conditions and the trading constraints.
2. If the trading constraints are infinite ($k_i = \infty$) this problem has the trivial solution consisting in immediate liquidation ($Q_1 = 0$) and the maximum value of the objective function is just ,…

$$ES_\alpha\left(\sum_{i=1}^{N} Q_i(0)\big(P_i(1) - P_i(0)\big)\right)$$

i.e. the expected shortfall for the portfolio over 1 day – a classical market risk measure

3. If the probability distribution of prices is a Gaussian `random walk, or Wiener process, (an assumption made by Almgren-Chriss, 2000) the optimization problem can be reduced to solving the quadratic programming problem:

$$Minimize \sum_{t=1}^{T-1}\sum_{ij=1}^{N} C_{ij}Q_i(t)Q_j(t)$$

over Q, T, with Q(0) given, subject to the constraints (1), where C_{ij} is the covariance matrix of 1-day price changes.

4. In the case N=1 the optimal liquidation strategy is

$$Q(t) = Q(0) - kt, \quad T = \frac{Q(0)}{k},$$

which means that liquidation of one asset portfolios should be done as fast as possible. We give a proof of this in the Gaussian case in the appendix, although we believe that

this is true for all (reasonable) probability distributions for stock returns.

2 Value Function and the Cost of Liquidity

We assume a risk-averse agent who uses $ES_\alpha(\min R(t))$ as his cost (or charge) for liquidity. The agent is interested in determining how much this charge should increase as a function of the initial position due to liquidity constraints.

The larger the position is, the more important liquidity risk becomes. This is incorporated in our model by assuming that only a fraction of daily volume can be transacted per day (which is the interpretation of the bound k in equation (1)).

To understand how size affects liquidity risk, at least in the limit of large quantities, we use a simple scaling argument. Assume N=1 and assume that $Q \coloneqq Q_1(0)$ is large with respect to $k := k_1$. Then, we have

$$\begin{aligned} R(t) &= \sum_{s=1}^{t-1} Q(s)\big(P(s+1) - P(s)\big) \\ &= \sum_{s=1}^{t-1} (Q - ks)\big(P(s+1) - P(s)\big) \\ &= \sum_{s=1}^{t-1} (Q - ks)\sigma\varepsilon_s \end{aligned}$$

where ε_s are i.i.d. random variables with mean zero and variance 1. Setting $T = \frac{Q}{k}$, we find that

$$\begin{aligned} R(t) &= \sigma k T^{3.2} \left(\frac{1}{\sqrt{T}} \sum_{s=1}^{t-1} \left(1 - \frac{s}{T}\right) \varepsilon_s \right) \\ &= \sigma \frac{Q^{1.5}}{k^{0.5}} \left(\frac{1}{\sqrt{T}} \sum_{s=1}^{t-1} \left(1 - \frac{s}{T}\right) \varepsilon_s \right). \end{aligned}$$

As $T \to \infty$, the term in parenthesis converges in distribution to the process

$$\int_{o}^{\tau}(1-u)d\,W_u, \qquad 0 \le \tau \le 1. \qquad (3)$$

Therefore, for large positions, the ES shortfall evaluated in the optimal strategy, which we interpret as the ``liquidity-adjusted cost'' should satisfy

$$ES_\alpha(\min_{t\le T} R(t)) \approx C_\alpha \sigma \frac{Q^{1.5}}{k^{0.5}}. \qquad (4)$$

Here C_α is a constant which depends only on the confidence level α, and which can be calculated explicitly based on the characterization of the asymptotic process (3).

In the rest of this paper, we will investigate the minimization problem for 1-stock portfolios using empirical probability distribution functions for returns, and see whether we find results which are consistent with (2).

Remark: A seemingly trivial, but nevertheless important, remark is that if we interpret (4) as a liquidity charge or liquidity-dependent risk measure, then it should be a non-linear function of position size. In practice, risk-managers might conduct polls to determine what the bid/ask spread should be for different sizes. This method gives a more mathematical approach to the problem of setting bid/ask spreads as a function of trade size.

3 Numerical Experiments

In the real world, stock prices are not Gaussian, and the optimization problem (2) generally with not have a closed-form solution. Assuming that the optimal liquidation is linear and uses the maximum speed allowed, we shall evaluate the expected shortfall upon liquidation for stocks based on Monte Carlo simulation. The goal is to evaluate the functional form of the cost function as a function of trade size.

For each simulation, we use historical bootstrapping to generate stock price returns and optimal linear liquidation. We then estimate the expected shortfall by averaging the worst $(1-\alpha) * 5000$ losses of 5000 simulations of stock price trajectories.

3.1 Historical Bootstrapping

Bootstrapping is a statistic method for evaluating functions (such as expectations, etc) for distributions which do not admit a closed-form solution (Efron and Tibshirani (1993)). There are no parametric assumptions for the underlying data in the bootstrapping method. The only assumption is that future paths will have the same basic historical return realizations that have been experienced in the past. So by creating a sampling distribution from past returns and drawing returns from the series randomly, we can create new future paths.

In our model, from 503 trading days' data, we created 502 days' return for 20 stocks stored by a 20-by-502 matrix. Here, daily closing price of stocks is used and we calculate the return as $r = ln\frac{S_t}{S_{t-1}}$ assuming continuously compound return. A key step in historical bootstrapping is to randomly select a cross sectional vector (a column) from the 20-by-502 return matrix, which means select a day from the 502-day historical window and use that day's returns (of 20 stocks). We will randomly select a column T* times from our matrix to get future stock returns for T* trading days. Next, we apply the selected returns to today's actual prices and this will simulate tomorrow's stock prices for 20 stocks. In our simulation, we generated 5000 future paths for each of the 20 stocks. So we will redo the step 4999 times to create 5000 total paths.

3 Numerical Results

Based on above considerations, we test whether the asymptotic result (2) gives insight on how the solution of the optimization problem behaves for real stocks as a function of trade size. We picked 20 stocks[2] and used historical bootstrapping to calculate their$ES_{0.99}$.

With a better sense of what the empirical solution will be like, we want to fit the expected shortfall of the portfolio PNL to a higher degree function of Q, so we take natural logarithm of initial share amount of the stock and compare to natural logarithm of $ES_{0.99}$.

[2] The 20 stocks are: SPY, IWM, GDX, UVXY, VTI, VWO, SIL, TLT, AGG, VOO, GOOG, YHOO, LNKD, AAPL, MET, BRK-B, AIG, BBY, MSFT, BBRY.

Figure 2 below is a linear regression of natural log of expected shortfall on the natural log of initial share amount using SPY data. To simplify the model, here we choose daily limit to be 1 share (or 1 ``lot') and simulate with initial quantity 1, 2, 3,...27(so trading time period will be up to 27 days). Then we can write an empirical solution to (8) is

$$ES_{0.99} = -2.0817 \cdot \sigma_{SPY} \cdot \frac{Q^{1.3489}}{\kappa^{0.5}} \tag{5}$$

By assuming daily limit to be 1 share, regressing natural log of $-ES_{0.01}$ on natural log of Q, and analyzing past 2 years' daily price of 20 stocks, we studied the degree of expected shortfall as a function of Q as shown in table 1. We noticed even though the power of Q is not exactly 1.5, but they are all between 1.2-1.4, which means under linear liquidation the expected losses follows an exponent p=1.2-1.4. Not only that the larger the initial share amount we have, the more risk we take, but also that the risk grows

Table 1: Empirical fitting of a power law for $ES_{0.99}$ as a function of Q

with a speed of 1.3489 power of Q (in the case of SPY). If we had an infinite Q, the result should be 1.5 because by Central Limit Theorem. The interpretation of this result is that the market will

Ticker	BBRY	GOOG	AAPL	BBY	MSFT
Power of Q	1.235	1.254	1.267	1.267	1.291
Ticker	AIG	LNKD	UVXY	MET	VOO
Power of Q	1.297	1.307	1.316	1.328	1.333
Ticker	SIL	AGG	VTI	SPY	VWO
Power of Q	1.338	1.339	1.342	1.349	1.351
Ticker	YHOO	IWM	BRK-B	GDX	TLT
Power of Q	1.351	1.37	1.389	1.39	1.391

use "curves" based on Q to the power of 1.2 to 1.5 to create bid/ask spreads for block trades in which the quantities are large. This is actually consistent with some empirical results coming from polls taken recently among traders in OTC market for swaps, which will be published elsewhere.

Because we assumed daily trading limit to be 1 share/day, it is easy to get the constants for $ES_{0.99}$ functions. Table 2 displays the constants and volatilities of 20 stocks and we noticed that the constants and volatilities are in fact positively correlated.

Previously, we didn't really study the role of κ played in $ES_{0.99}$ as we assume it to be 1. The theoretical solution to (8) has terms of daily limit κ to the power of 0.5. Next, we want to test it by picking IWM[3] and using its average trading volume of 36,852,800

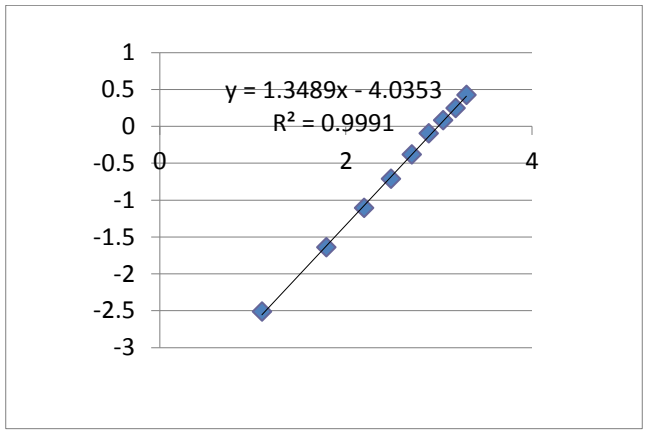

Figure 2: (SPY) linear regression of $\ln(ES_{0.99})$ on lnQ with daily limit 1

from Yahoo! Finance. After doing similar regression on Q, we have a solution to (8):

$$ES_{0.99} = -2.6074 \cdot \sigma_{IWM} \cdot \frac{Q^{1.2939}}{\kappa^{0.2827}} \tag{6}$$

-2.6074 is calculated using daily limit 1, 0.2827 is calculated using Goal Seek function in Excel and 1.2939 suggests the linear regression of $\ln(ES_{0.99})$ on lnQ has slope 1.2939. Because the daily trading volume now is huge, if the portfolio is traded longer, we suspect that the degree of ES (as a function of Q) might be higher as time would put more weight on losses. So we analyzed the expected shortfall of up to 5 trading days versus the expected shortfall of 10-day trading period to 35-day trading period. The

[3] IWM/iShares Russell 2000 Index is an ETF.

results as shown in Figure 3 suggests that ES as a function of Q has a lower degree when trading period is short and has nearly a degree of 1.4 if liquidation takes more than 10 days.

All of the results in this section are obtained assuming $Q_s = Q - \kappa s$.

Ticker	IWM	SPY	GDX	UVXY	VTI
C	2.6079	2.0818	5.5070	18.4562	2.1688
Volatility	1.119%	0.849%	2.454%	17.860%	0.875%
Ticker	VWO	SIL	TLT	AGG	VOO
C	3.3103	7.0630	2.1020	0.6232	2.1241
Volatility	1.304%	2.540%	0.900%	0.203%	0.850%
Ticker	GOOG	YHOO	LNKD	AAPL	MET
C	4.8194	3.3715	6.8074	6.4197	4.4046
Volatility	0.014%	1.561%	2.766%	1.834%	1.807%
Ticker	BRK-B	AIG	BBY	MSFT	BBRY
C	1.82890	4.97093	9.72763	5.07303	16.55393
Volatility	1.770%	0.885%	3.050%	1.454%	4.237%

Table 2: constant in $ES_{0.99}$ as a function of Q for 20 stocks

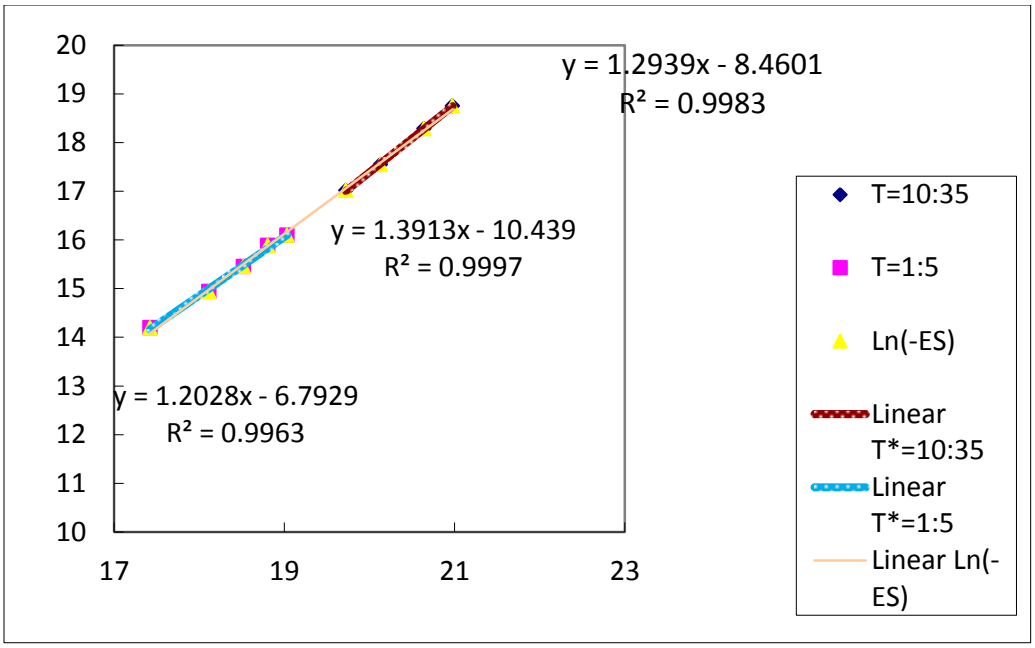

Figure 3: (IWM) linear regression of $\ln(ES_{0.01})$ on lnQ with daily limit 36,852,800

4 Conclusions

In this paper, we extend the classical portfolio risk measures by assuming that positions have to be liquidated over time with a volume constraint. In our model, we use Expected Shortfall for the **worst drawdown** during the liquidation period

Theoretically, the expected shortfall for Gaussian model should follow a 1.5 power-law function of initial share amount. By running data of 20 different stocks and different initial amounts, we observed empirically power-law dependence on Q with powers between 1.2 to 1.4. For moderate quantities, the power tends to be smaller and when the trading size gets larger, the function of ES tends to be similar to the theoretical solution. This is not surprising because 1.5 should be the size for large volume from the Central Limit Theorem and the argument presented in Section 2.

This model can be extended in several directions, such as adding more dimensions to our portfolio. If different stocks in the portfolios have different T* (time to liquidate), the optimal strategy may not be the one which liquidates at maximum speed. In

particular, we believe that N>2 this will give rise to more complex liquidation strategies in which trading will aim at hedging market exposure by (i) hedging illiquid positions with liquid products and (ii) synchronizing the liquidation of long and short positions

5 Appendixes

5.a Figures

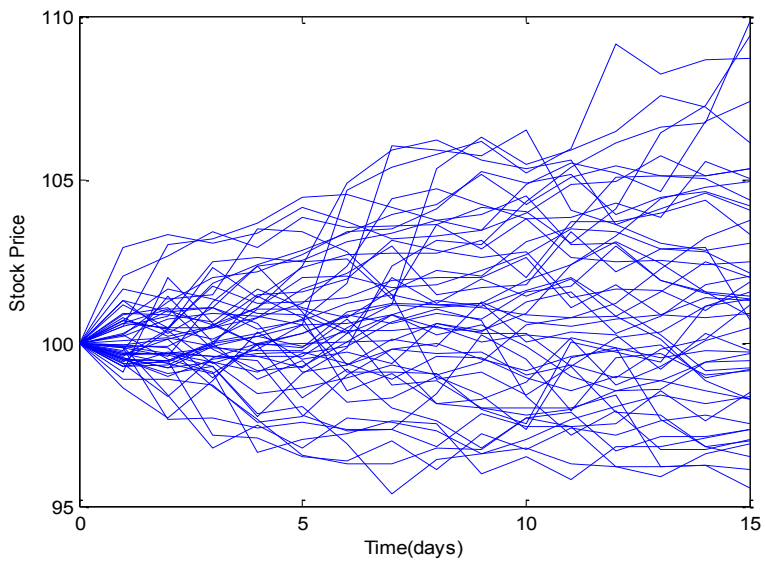

Figure 4: Historical simulation of 50 paths of stock price for 15-day period with SPY's data that σ =0.00849299 and $S_0 = 100$.

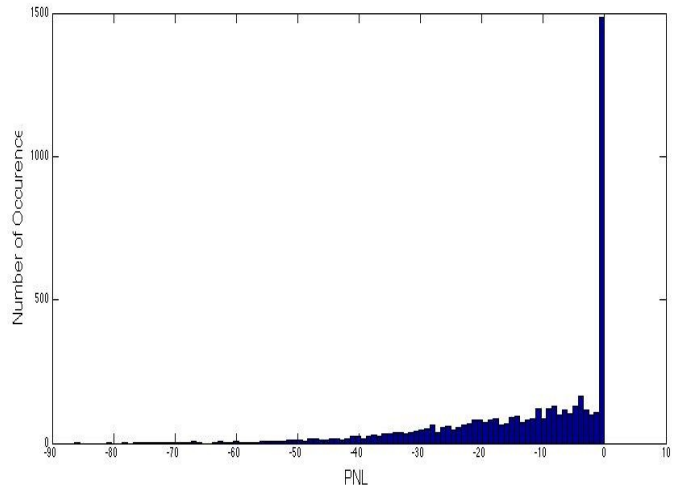

Figure 5: Histogram of minimizing portfolio PNL over T*-day trading period with initial shares 1500, daily trading limit 100 shares and initial stock price 100 simulated 5000 times

5.b Optimal Solution of Minimizing PNL with Limit

In this section, we will give the proof that $Q - \kappa s$ is the optimal liquidation solution given a trading limit. First we want to show, if the stock return follows Gaussian distribution then the worst drawdown is proportional to the integral of portfolio variance. This follows the properties of ES for Gaussians and the Brownian reflection principle:

In a continuous time, as the quantity corresponds to the worst loss one can have, we minimize portfolio PNL as following:

$$ES_{\alpha}(\min_{t \leq T*} \int_0^t Q_s dX_s) \tag{7}$$

If the stock returns are Gaussian then we have

$$dX_t = \sigma dW_t = \sigma \varepsilon \sqrt{\Delta T} \tag{8}$$

ε is a random draw from standardized normal distribution.

Then equation (19) becomes a problem of finding ES for minimum of a Brownian motion. From reflection principle, the minimum of a Brownian motion is the absolute value of a normal random variable. So the problem is equivalent to minimizing variance under constraints:

$$ES_\alpha(-|N(0,\sigma^2)|) = ES_\alpha(-\sigma^2 \cdot N(0,1)) \\ = \sigma \cdot ES_\alpha(-N(0,1)) \tag{9}$$

So the ES_α of the worst drawdown is proportional to the integral of the portfolio variance,

$$C_\alpha \sqrt{\min_{\substack{Q(0)=Q \\ Q(T)=0 \\ |\dot{Q}|\leq\kappa \\ T}} \sigma^2 \int_o^T {Q_s}^2 ds} \tag{10}$$

C_α is a constant corresponding to loss at level α.

Next, we will show $Q - \kappa s$ is the optimal liquidation solution to problem (10).

Lemma 1: If $\left|R(\dot{t})\right| \leq \kappa$, and $R(0) = Q$, then $R(t) \geq Q - \kappa t$.

Proof: $R(\dot{t}) \geq -\kappa \implies R(t) - R(0) = \int_0^t \dot{R} \geq -\kappa t \implies R(t) \geq Q - \kappa t$.

Lemma 2: If $R(t) \geq Q - \kappa t$, then $\int_0^t \sigma^2 R(t)^2 \geq \int_0^t \sigma^2 (Q - \kappa t)^2$.

Proof: Obvious.

Now we can conclude $Q - \kappa t$ is the optimal solution to problem (10).

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