1. **Ito and Wiener integrals**

   a) Let \( f(t) = \sin t \). Consider the Ito integral \( X = \int_0^{\pi/2} \sin(t) \, dw \). Calculate the mean and the variance of \( X \).

   b) Performing a formal integration by parts and write down the "Wiener version" of this integral. Let us denote it by \( Y \).

   c) Simulate 3000 realizations of \( X \) and \( Y \) by discretizing the integrals with \( dt = 0.001 \). Compare the corresponding histograms.

2. **First-passage times**

   a) Let \( x(t) \) be a Brownian motion and \( t_0 \) a fixed time. Check that \( y(t) = x(t + t_0) - x(t_0) \) is a Brownian motion and \( y \) is independent of \( x(t), t \leq t_0 \).

   b) Let \( \tau_a = \min\{t : x(t) = a\} \) be the first time when \( x(t) \) reaches the value \( a \). Argue that \( x(t + \tau_a) - x(\tau_a) \) is a new Brownian motion, independent of \( x(t), t \leq \tau_a \). (No rigorous proof needed!).

   c) The following is known as the reflection principle for Brownian motion: if \( a \) and \( b > 0 \), then,

   \[
   P(\max_{0 \leq s \leq T} x(s) > a, x(T) < a - b) = P(x(T) > a + b)
   \]

   Use part b) to justify this formula, noting that \( y(t) \) and \( -y(t) \) have the same distribution. Make a picture and explain why it is called "reflection principle".

   d) Compute \( P(\max_{0 \leq s \leq T} x(s) > a, x(T) < a) \) and use symmetry to get the following formula

   \[
   P(\max_{0 \leq s \leq T} x(s) > a) = 2P(x(T) > a) = \frac{2}{\sqrt{2\pi T}} \int_a^{\infty} e^{-\frac{x^2}{2T}} \, dx.
   \]

   What if \( a < 0 \)?
e) Explain why it is true that \( P[\max_{0\leq t \leq T} x(t) > a] = P[\tau_a \leq t] \).
Use your result in d) and the change of variables \( s = Ta^2 / x^2 \) to find the probability density function for the hitting time \( \tau_a \).

3. **Martingales, iterated conditional expectation.**

a) Let \( Y, X_1, X_2 \) be a random variable such that \( E|Y|^2 < \infty \). Show using the definition of conditional expectation that
\[
E[E(Y | X_1, X_2) | X_2] = E(Y | X_2)
\]
Give an intuitive explanation.

b) As before, let \( E|Y|^2 < \infty \) and let \( X_1, X_2, \ldots \) be an infinite sequence of random variables.
Define \( Z_n = E[Y | X_1, X_2, \ldots X_n] \)
Show that \( Z_n \) is a martingale, i.e. \( E[Z_n | Z_{n-1}] = Z_{n-1} \)
(To show that \( Z_n \) is a martingale you may want to justify and use the following three statements:
\[
E[Z_n | X_1, \ldots X_{n-1}] = Z_{n-1}, \quad E[Z_n | X_1, \ldots X_{n-1}, Z_{n-1}] = Z_{n-1} \text{ and}
E[E[Z_n | X_1, \ldots X_{n-1}, Z_{n-1}] | Z_{n-1}] = E[Z_n | Z_{n-1}]
\]
)

c) Let \( Y \) be uniformly distributed on \([0,1]\) and let \( X_n \) be defined by
\[
X_n = \begin{cases} 
0, & Y \in \left(\frac{j}{2^n}, \frac{j+1}{2^n}\right], \; j \text{ even} \\
1, & j \text{ odd}
\end{cases}
\]
Show that \( X_n \) are independent random variables.

d) Calculate \( E[Y | X_1, \ldots X_n] \) explicitly.

e) Check that \( E[Y | X_1, \ldots X_n] \to Y \) \( \to Y \)

f) Give a different example of a sequence \( X_n \) such that \( E[Y | X_1, \ldots X_n] \to \bar{Y} \)
and \( \bar{Y} \neq Y \).
g) Explain how this problem is relevant to problem #3 from problem set 3, assuming \( Y, X_1, \ldots X_n, \ldots \) are Gaussian.
4. **Martingales and Fokker-Plank equation**

Let $F(t) = F(w(t), t)$ be a function. Define a conditional expectation

$$E[F(w(t)) \mid w(s) = x] = f(x, s), \quad s < t.$$  

\[ F_{w(t)}(w(t)) = E[F(w(t)) \mid w(s) = x] = f(x, s), \quad s < t. \]

a) Show that $f(x, s) = \frac{1}{\sqrt{2\pi(t-s)}} \int F(y)e^{-\frac{(y-x)^2}{2(t-s)}} dy$.

b) Conclude that $\frac{\partial f}{\partial s} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0$. This is known as Fokker-Plank equation or Kolmogorov backward equation.

c) Use the definition of $f(x, s)$ and your result in b) to justify the following statement: “Let $\varphi(t, w(t))$ be a smooth function. It is a martingale if and only if it satisfies the Fokker-Plank equation, that is, if $\frac{\partial \varphi}{\partial s} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2} = 0$.”

e) Prove the statement using Ito’s formula.

f) Verify that $\varphi(x, t) = e^{\frac{\lambda}{2} x^2 t}$ is a solution of the Fokker-Plank equation. This gives you another way to show that $e^{\frac{\lambda}{2} (w(t) - \lambda t)}$ is a martingale.