1. Let's Make a Deal Paradox. This paradox is related to a popular television show in the 1970's. In the show, a contestant was given a choice of three doors of which one contained a prize. The other two doors contained gag gifts like a chicken or a donkey. After the contestant chose an initial door, the host of the show then revealed an empty door among the two unchosen doors, and asks the contestant if he or she would like to switch to the other unchosen door. The question is should the contestant switch. Do the odds of winning increase by switching to the remaining door?

2. Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ be sequences of independent standard $N(0,1)$ random variables. Define formally

$$\eta(t) = \sum_{n=-\infty}^{\infty} a_n \cos(2\pi nt) + b_n \sin(2\pi nt).$$

This object is known as white noise.

To approximate it set

$$\eta_N(t) = \sum_{n=-N}^{N} a_n \cos(2\pi nt) + b_n \sin(2\pi nt).$$

a) Show that

$$\langle \eta_N(t)\eta_N(t') \rangle = E[\eta_N(t)\eta_N(t')] = \frac{\sin 2\pi \left( N + \frac{1}{2} \right) (t-t')}{\sin \pi (t-t')}$$

b) Conclude that if $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ are disjoint intervals $(0 \leq a_1 < b_1 < a_2 < b_2 \leq 1)$ then

$$\lim_{N \to \infty} \int_{I_1} \int_{I_2} \eta_N(t) \eta_N(t') dt dt' = 0.$$

c) Now prove that for any interval $I \in [0,1]$

$$\lim_{N \to \infty} < \int_I \eta_N(t)^2 > = |I|,$$

where $|I|$ is the length of the interval.

Hint: At some point you will end up with the following integral
\[ \int \int_{I'} \frac{\sin 2\pi \left( N + \frac{1}{2} \right)(t-t')}{\sin \pi(t-t')} \, dt \, dt' . \]

You may want to prove that for any interval \( I \in [0,1] \) and \( t' \in I \)

\[ \int_{I'} D_N(t-t') \, dt = \int_{I'} \frac{\sin 2\pi \left( N + \frac{1}{2} \right)(t-t')}{\sin \pi(t-t')} \, dt \xrightarrow{N \to \infty} 1 , \]

where \( D_N(t) \) is known as the Dirichlet kernel. The proof can be conducted in two steps.

First, prove that \( \int_0^1 D_N(t-t') \, dt = 1 \). For this recall from a) that \( D_N \) is actually a sum of cosines and integrate this sum.

Second, prove that \( \lim_{N \to \infty} \int_{\delta \leq |t-t'| \leq 1} D_N(t-t') \, dt = 0 \). In this case the denominator is bounded away from zero and the statement can be proved by integrating by parts. This shows you that the value of the integral is “concentrated in a small neighborhood of \( t' \)”. Combine the two steps to conclude that

\[ \int_{I'} D_N(t-t') \, dt \to 1 \text{ if } t' \in I . \]

d) Use Excel (Matlab or other software) to plot a graph of \( \eta_N(t) \) for \( N = 125 \). This should look like “white noise”.

e) For the plotted realization of \( \eta(t) \) compute the variance. Compare it with the theoretical value.

3. Define \( X = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \), where \( x_n \)'s are independent identically distributed random variables that take values 0 or 1 with probability \( \frac{1}{2} \). Show that \( X \) is uniformly distributed.