1. Let $x(t)$ be a Brownian motion and let $\varphi(x,t) = E[e^{-\int_0^t x^2 \, ds} \mid x_0 = x]$.

a) Use the Feynman-Kac formula to show that $\varphi$ satisfies

$$\left\{ \begin{array}{l}
\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} - x^2 \varphi = 0, \quad t < T \\
\varphi(x,T) = 1
\end{array} \right.$$ 

b) Using the ansatz $\varphi(x,t) = e^{a(t)x^2 + b(t)}$ derive, and then solve a system of ODE’s for $a(t)$ and $b(t)$.

c) Calculate exact numerical answer for $\varphi(0,0) = E[e^{-\int_0^T x^2 \, ds} \mid x_0 = x]$.

2. Let $y_i$ be Ornstein-Uhlenbeck process

$$dy_i = -\frac{k}{2} y_i dt + dw_i.$$ 

a) Show that $r_i = \frac{\alpha^2}{4} y_i^2$ satisfies the stochastic differential equation

$$dr = \alpha \sqrt{r} \, dw + k(\theta - r)dt$$ 

for some $\theta$. Calculate $\theta$ as a function of $\alpha$ and $k$.

3. As in Problem 2, but now there are $n$ Ornstein-Uhlenbeck processes

$$dy_i = -\frac{k}{2} y_i dt + dw_i,$$ 

where $w_i$ are independent Brownian motions.

a) Define $r = \frac{\alpha^2}{4} \sum y_i^2$. Show that $r$ satisfies

$$dr = \alpha \sqrt{r} \, dw + k(\theta_n - r)dt.$$
Give a formula for $\theta_n = \theta_n(\alpha, k)$.

b) Show intuitively that if $n = 1$ the process $r(t)$ hits 0, but if $n > 1$ $r(t)$ never hits zero. (Hint: for small $r$ the process looks like the square of radius of Brownian motion. Use transience and recurrence of $n$-dimensional Brownian motion).

c) Check that the distribution function of $r(t)$ is a non-central $\chi^2$. With how many degrees of freedom?


Assume that the interest rate follows the Ornstein-Uhlenbeck process

$$dr = -k(r_n - r)dt + \sigma dw$$

a) Write down exact solution for $r(t)$.

b) Write a linear regression model

$$r(t_{n+1}) = a + br(t_n) + \varepsilon$$

representing the relation between $r(t_{n+1})$ and $r(t_n)$.

Use your result in a) to give theoretical values for $a$, $b$ and $\sigma^2 = \text{var}(\varepsilon)$.

c) Run the above regression to estimate $a$, $b$ and $\sigma^2$. Using estimated values solve for $r_\infty$, $k$ and $\sigma$. 