1. **The Ornstein-Uhlenbeck process**

Let \( x(t) \) be given by the following stochastic differential equation:

\[
dx = (\alpha_1 + \alpha_2 x(t))dt + \alpha_3 dw(s)
\]

a) Write an explicit solution for \( x(t) \) in terms of \( w(.) \). [Hint: method of variation of parameters in ODE.]

b) Find the mean and the variance of \( x(t) \) and characterize its distribution.

c) In the case \( \alpha_2 < 0 \) show that the distribution of \( x(t) \) converges to a stationary distribution. Give the parameters of the stationary distribution. In this case, the parameter \( \kappa = -\alpha_2 \) is known as the mean-reversion constant. What can you say about \( \alpha_2 > 0 \)?

d) Compute the structure function \( \text{var}(x(t + \delta t) - x(t)) \)
   (i) from an arbitrary starting point and
   (ii) assuming the equilibrium distribution.

   How does it behave for large \( \delta t \)? Compare with the structure function for Brownian motion. Compute also the two-point correlation function \( \text{Corr}(x(t + \delta t), x(t)) \).

e) Compute the moment-generating function \( E e^{\lambda \int_0^t x(s)ds} \).

2. Download the series of the VXO CBOE volatility index and of the S&P-500 index (^GSPC) for 5 years.

   a) Define a process \( y(t) = aw_1(t) + bx(t) \), where

   \[
dx = \kappa(x - x_0)dt + \sigma dw_2
   \]

   is the Ornstein-Uhlenbeck process defined in problem 1, and \( w_1 \) is independent Brownian motion. Calculate explicitly the structure function for \( y \).

   b) Using the data, construct an estimator for the structure function. Hint: use lagged differences with lag=1,2,3,…., days going backward and compute the empirical variances of the differenced processes.

   c) Use your estimator of the structure function to estimate the parameters of the process \( y(t) \). Describe, if any, structural differences between the S&P cash
Consider the following stochastic volatility model:

\[
\begin{align*}
\frac{dx(t)}{\sigma(t)} & = dw(t) \\
\frac{d\sigma(t)}{\sigma(t)} & = k\sigma(t)dz(t)
\end{align*}
\]

where \( z \) and \( w \) are two Brownian motions with \( E(dwdz) = \rho dt \), \( x(0) = 0, \sigma(0) = \sigma_0 \).

In this problem, you will use most of your knowledge of Ito calculus to give explicit formulas for \( E(x(t)) \), \( E(x^2(t)) \) and \( E(x^3(t)) \).

a) Compute \( E(x(t)) \).

b) Show that \( \sigma(t) = \sigma_0 e^{\frac{kz(t)-\frac{1}{2}t^2}{2}} \).

c) Apply Ito’s formula to \( f(x) = x^2 \) to get

\[
x^2(t) = \int_0^t 2x(s)\sigma(s)dw(s) + \int_0^t \sigma^2(s)ds
\]

Take expectation of both sides of this equation.

d) In order to complete the computation of \( E(x^2(t)) \) you will need to compute \( \int E[e^{2z(s)-k^2s}]ds \). Write \( e^{2z(s)-k^2s} = e^{k^2s}e^{2z(s)-2k^2s} \) and use your knowledge of exponential martingales.

Give an explicit formula for \( E(x^2(t)) \).

e) Apply Ito’s lemma to \( f(x) = x^3 \) to show that

\[
E(x^3(t)) = E\left[\int_0^t \sigma^2(s)\int_0^s \sigma(u)dw(u)udu\right]
\]

f) Write \( dw \) as \( adz + bd\tilde{z} \) where \( \tilde{z} \) is a Brownian motion independent of \( z \). Find the coefficients \( a \) and \( b \).

Rewrite \( E(x^3(t)) \) as the sum of \( dz \) and \( d\tilde{z} \) integral. Argue that the expectation of \( d\tilde{z} \) integral is zero.

g) Integrate back \( \sigma(u)dz(u) \). Use the technique from d) to give explicit formula for \( E(x^3(t)) \).