Stochastic Calculus: Lecture 3

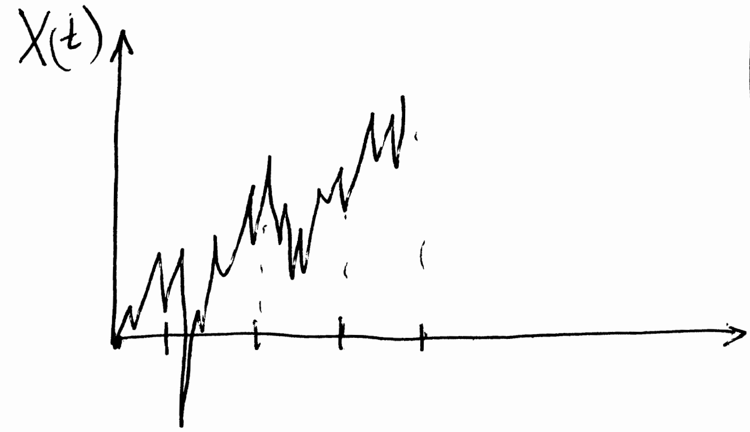
Brownian Motion: (formal definition in terms of statistics)

1. Gaussian process $\left((X(t_1) \dots X(t_n)) \text{ Gaussian } \forall (t_1 \dots t_n) \right)$

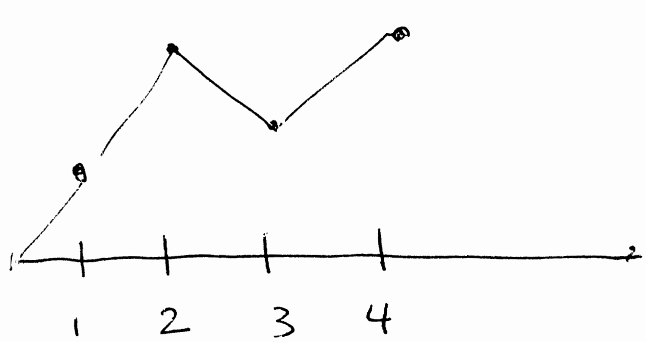
2. $X(t+\delta t) - X(t)$, $X(t)$ independent

3. $X(0) = 0$, $X(t) \sim N(0,t)$

B.M. = a continuous-time random walk with Gaussian increments!



How can we construct such function?



If we look on a set of discrete times, we have a standard r.w

$$\tilde{X}_n = X(n)$$

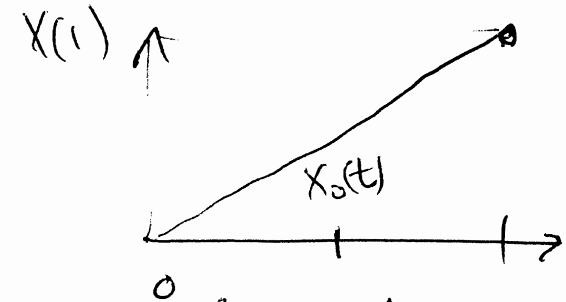
$$\boxed{\tilde{X}_{n+1} = \tilde{X}_n + \nu_{n+1} \qquad \nu_{n+1} \sim N(0,1)}$$

Construction of BM by interpolation (A.N. Kolmogorov).

Consider $0 < t < 1$. $X(t) =$ BM (finally)

step 1. $X(1) \sim N(0,1)$.

$$X_0(t) = t\,X(1)$$



step 2. The next step is to refine this function by considering $(0, \tfrac{1}{2}, 1)$.

$$X(\tfrac{1}{2}) = \beta\, X(1) + \xi_1$$

← regression of $X(\tfrac{1}{2})$ on $X(1)$ (best predictor)

$$E(\xi_1) = 0.$$

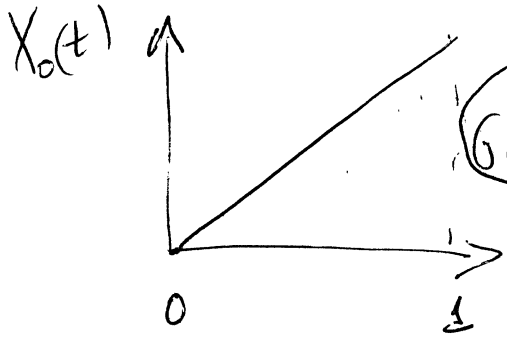
$$E\left[X(\tfrac{1}{2})\,X(1)\right] = E\,X(\tfrac{1}{2})^2 + E\,X(\tfrac{1}{2})\left(X(1) - X(\tfrac{1}{2})\right)$$

$$= \frac{1}{2}$$

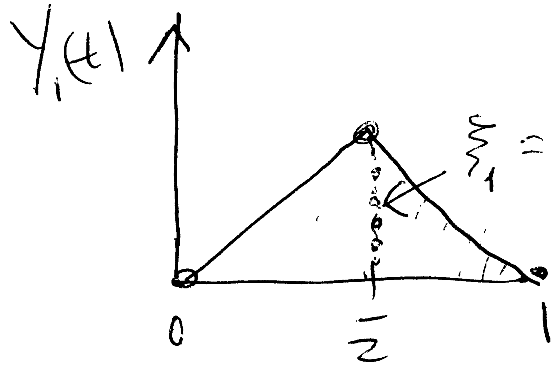
$$\frac{1}{2} = \beta \quad \therefore \qquad X(\tfrac{1}{2}) = \frac{1}{2} X(1) + \xi_1$$

$$E(\xi_1) = 0 \qquad E\,\xi_1^2 + \beta^2 = \frac{1}{2} \qquad \boxed{E\,\xi_1^2 = \frac{1}{4}}$$

<u>Define</u>: $X_1(t) = X_0(t) + Y_1(t)$



$\sigma_0 = 1$

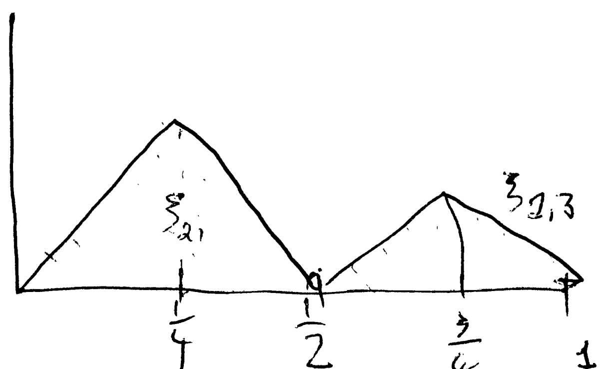


$E\xi_{11}^2 = \frac{1}{4}$

$\sigma_1 = \frac{1}{2}$

$X_1(t)$ is consistent with the statistical properties of BM on the set of times $(0, \frac{1}{2}, 1)$.

<u>Step 3</u>: Refine each "lobe" of $Y_1(t)$



$E\xi_{2,j}^2 = \sigma_2^2$

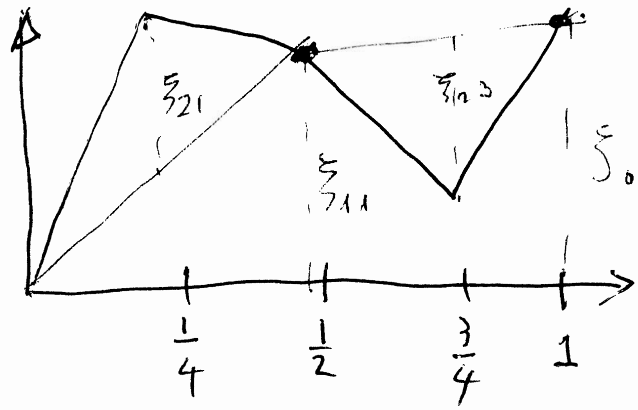
$\sigma_2 = \frac{1}{\sqrt{8}}$

$X(\frac{1}{4}) = \beta\, X(\frac{1}{2}) + \xi_{21}$

$\frac{1}{4} = \beta\, \frac{1}{2} + 0 \quad \therefore \quad \beta = \frac{1}{2}$

$\frac{1}{4} = \frac{\beta^2}{2} + E\xi_{21}^2 \qquad \frac{1}{4} = \frac{1}{8} + \sigma_2^2 \qquad \sigma_2 = \frac{1}{2^{3/2}}$

$$X_2(t) = X_0(t) + Y_1(t) + Y_2(t)$$

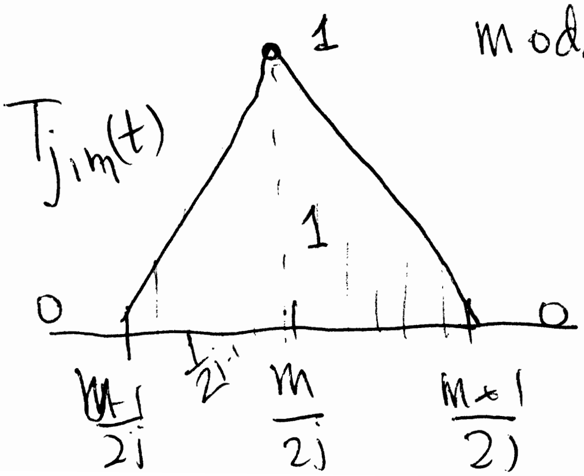


$X_2(t)$ is BM restricted to $\left(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$

Step 4: Define the refinement process inductively for all scales!

$$X_N(t) = X_0(t) + \sum_{j=1}^{N} Y_j(t)$$

$$Y_j(t) = \sum_{\substack{m=1 \\ m \text{ odd}}}^{2^j-1} \xi_{j,m} \, T_{j,m}(t) \leftarrow \text{tent function}$$



$$E\,\xi_{j,m}^2 = \sigma_m^2 \, ;$$

$$\sigma_m = \frac{1}{2^{\frac{j+1}{2}}}$$

Proposition:

Brownian Motion is the limit of $X_N(t)$ as $N \to \infty$. This limit exists with probability $= 1$ and defines a continuous path.

In fact, if $\|Y_j\|_\infty = \max_{t \in [0,1)} |Y_j(t)|$

$$P\left[\|Y_j\|_\infty > a\right] \cancel{=} P\left[\max_{1 \le m < 2^j - 1} |\xi_{j,m}| > a\right]$$

$$\le 2^j P\left[|\xi_{j,1}| > a\right]$$

$$\le 2^j \frac{1}{a^4} 3\left(\frac{1}{2^{j+1}}\right)^2$$

$$= \frac{3}{a^4 2^{j+2}} \le \frac{1}{a^4 2^j}.$$

$\boxed{\left(E N^4 = 3\sigma^2\right) \text{ Need Gaussian } 4^{th} \text{ Moment!}}$

If $a = \frac{1}{2^{j\varepsilon}}$)

$$P\left[\|Y_j\|_\infty > \frac{1}{2^{j\varepsilon}}\right] < \frac{2^{4j\varepsilon}}{2^j} = \frac{1}{2^{j[1-4\varepsilon]}}$$

If $\varepsilon < 1/4$

$$P\left[\|Y_j\|_\infty > \left(\frac{1}{2^\varepsilon}\right)^j \right] \leq \left(\frac{1}{2^{1-4\varepsilon}}\right)^j$$

$$\sum_{j=1}^{\infty} P\left[\|Y_j\|_\infty > \left(\frac{1}{2^\varepsilon}\right)^j\right] < \infty$$

The convergence of this series implies, by the Borel-Cantelli Lemma of elementary Probability, that

$$P\left\{ \|Y_j\|_\infty \leq \left(\frac{1}{2^\varepsilon}\right)^j \text{ for } j > j^*(\xi) \right\} = 1.$$

where $j^*(\xi)$ depends on the sequence

$\xi_0, \xi_{11}, \xi_{21}, \xi_{23}, \xi_{31}, \xi_{33}, \xi_{35}, \xi_{37}, \ldots$

.... Since, eventually, $\|Y_j\|_\infty < \left(\frac{1}{2^\varepsilon}\right)^j$ we have

$$\sum_j \|Y_j\|_\infty < \infty \quad \left[\text{for almost all sequences } \xi\right]$$

This convergence of the norms ⑦
implies ~~that~~ $\sum Y_j(t)$ is a
continuous function.

<u>Note</u>: This is a sketch of the
proof of continuity of Brownian
paths. It "works" ~~because~~ (mostly)
because

$$E\,\xi_{j,m}^4 = \frac{3}{2^{2(j+1)}} \ll \frac{1}{2^j} \quad \text{as } j \to \infty$$

recalling that $\xi_{j\,m} =$

$$X\left(\frac{m}{2^j}\right) - \frac{1}{2}\left(X\left(\frac{m-1}{2^j}\right) + X\left(\frac{m+1}{2^j}\right)\right), \text{ what}$$

we are saying is that

$$E\left(X(t+\delta t) - X(t)\right)^4 = 3\,\delta t^2 \ll \delta t.$$

$\delta t \to 0$.

Kolmogorov showed, using a
similar construction, that if

a process has a distribution function such that

$$E|X(t+\delta t) - X(t)|^b \le C|\delta t|^{1+a}$$

with $a, b, c > 0$, then $\underline{X(t)}$ ~~has~~ can be constructed as a continuous function of t (just like we constructed BM).

Example: Revisiting AR(1)....

$$X_{n+1} = b X_n + \nu_{n+1}$$

$$\nu_{n+1} = N(0, \sigma^2)$$

$$b < 1$$

$$E X_n = 0 \; ; \quad E X_n^2 = \frac{\sigma^2}{1-b^2} = \bar{\sigma}^2$$

$$X_n = b^n X_0 + \sum_{0}^{n-1} b^{n-j} \nu_j$$

$$\sigma^2(X_n) = \sigma^2(1 + b^2 + b^4 + \dots + b^{2n}) = \frac{\sigma^2}{1-b^2}$$

$$E(X_n \cdot X_m) = b^{n-m} \, E X_m^2$$

$$\mathrm{Corr}(X_n, X_m) = b^{n-m}$$

This suggests defining a gaussian process in continuous time, $X(t)$, such that

$$\cancel{E\left[X(t+\delta t)\cdot X(t)\right] = e^{-k\delta t}\sigma^2}$$

$$\begin{cases} E\, X(t) = 0 \\ E\left[X(t+\delta t)\cdot X(t)\right] = \sigma^2 e^{-k\cdot \delta t} \end{cases}$$

If such process existed then

$$E\left[X(t+\delta t) - X(t)\right]^2 =$$

$$E\left(X(t+\delta t)\right)^2 + E\left(X(t)\right)^2 - 2E\,X(t)X(t+\delta t)$$

$$= \sigma^2 + \sigma^2 - 2\sigma^2 e^{-k\delta t}$$

$$= 2\sigma^2\left(1 - e^{-k\delta t}\right)$$

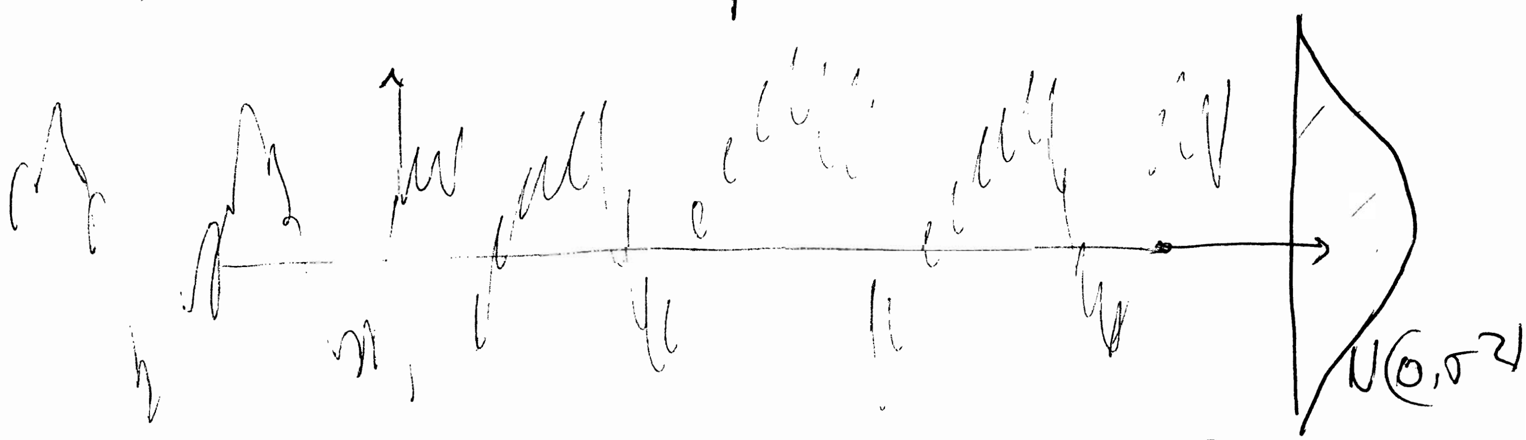
$$X(t+\delta t) - X(t) \sim N\left(0, 2\sigma^2(1-e^{-k\delta t})\right)$$

$$\therefore \quad E\left(X(t+\delta t) - X(t)\right)^4 \cong 3\cdot 4\sigma^4(1-e^{-k\delta t})^2$$

$$= 12\sigma^4(1-e^{-k\delta t})^2$$

$$\underset{\delta t \ll 1}{\cong} 12\sigma^2 K^2(\delta t)^2$$

Thus, the AR(1) process admits a version in continuous time which has continuous paths that look like



This process is ~~the~~ called the Ornstein-Uhlenbeck process. We will study it later in detail.

Further study of BM.

1. Let $X(t)$ be BM.

$$\boxed{\mathrm{Cov}(X(t), X(t+\delta t)) = t} \qquad (\delta t > 0)$$

$$\mathrm{Cov}(X(t_1), X(t_2)) = \min(t_1, t_2)$$

Notice that this is not a function of $t_2 - t_1$, BM is not stationary (Ornstein-Uhlenbeck is).

2. For any t and any $t_1 < \dots < t_n < t$,

$$(*) \quad E[X(t) \mid X(t_1), \dots X(t_n)] = \cancel{A}\bar{X}(t_n)$$

Proof $X(t) = X(t_n) + \nu$

where ν is independent of $X(t_n)$

Best linear predictor is $X(t_n)$, but since BM is Gaussian $(*)$ holds.

Conditional expectation relative to the past until time θ ($\theta < t$).

$\mathbb{E}[X(t) \mid X(s),\ s \le \theta]$ is the best predictor (in the sense of least squares) of $X(t)$ given $X(s)$, $s \le \theta$.

$$\mathbb{E}[X(t) \mid X(s),\ s \le \theta] = \underbrace{\varphi[\cancel{X(\theta),\ \theta \le t_n}]}_{Y}$$

$\Longleftrightarrow$ if

$$\boxed{E\,|X(t) - Y|^2 \le E\left[\,|X(t) - f(X(t_1) \ldots X(t_n))|^2\right]}$$

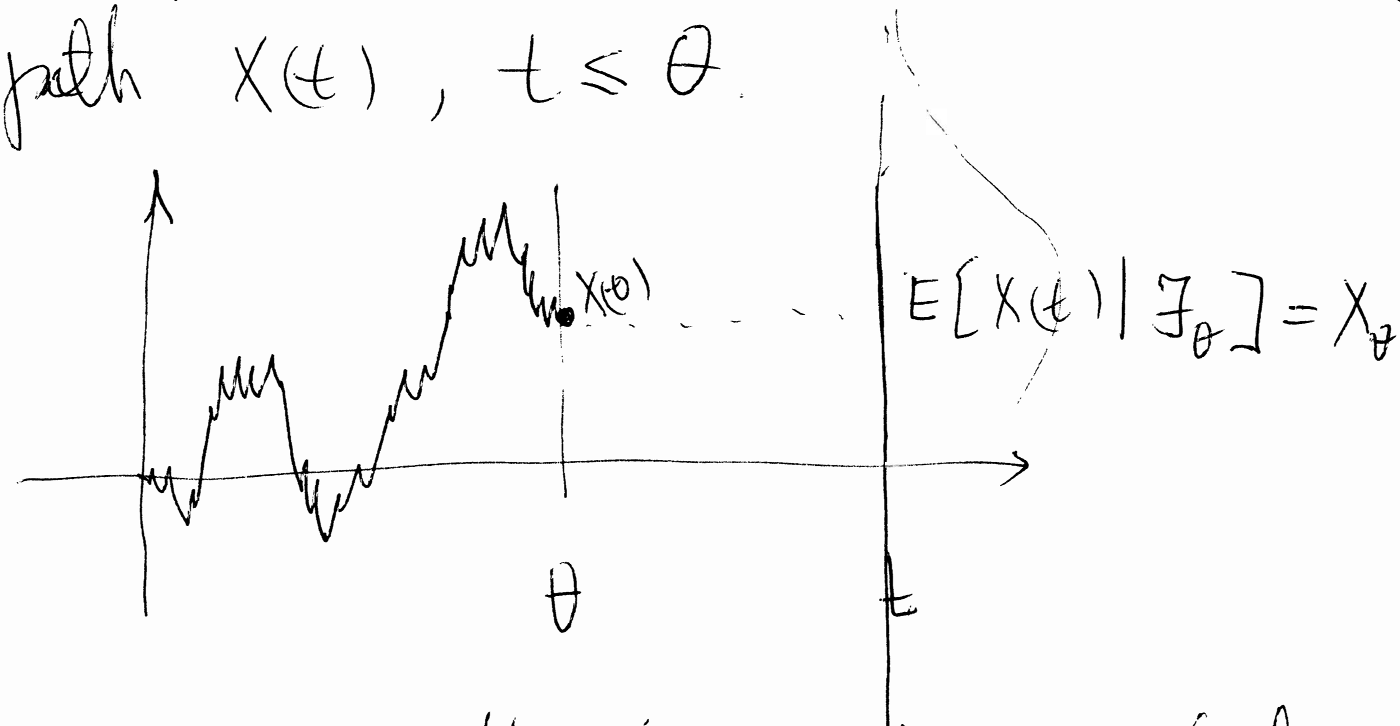
for all f and all $t_1 < t_2 \ldots < t_n \le \theta$.

We sometimes write

$$E[X(t) \mid X(s), \ldots s \le \theta] =$$

$$E[X(t) \mid \mathcal{F}_\theta]$$

The idea of $E(\xi \mid \mathcal{F}_\theta)$ is to predict ξ based on the path $X(t)$, $t \leq \theta$.



A process with the property that $E\left(X(t) \,\middle|\, \mathcal{F}_\theta\right) = X_\theta$ is called a <u>Martingale</u>.

Conditional distribution of BM

$$\mathbb{E}\left[e^{ikX(t)} \,\middle|\, \mathcal{F}_s\right] = e^{ikX(s)} \cdot \mathbb{E}\left(e^{ik(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right)$$

$$= e^{ikX(s)} \, e^{-\frac{k^2}{2}(t-s)}$$

Also, for real exponentials $(s<t)$

$$E\left[e^{\lambda X(t)} \mid \mathcal{F}_s\right] = e^{\lambda X(s)} e^{\frac{1}{2}\lambda^2(t-s)}$$

$$\therefore \quad E\left(e^{\lambda X(t) - \frac{1}{2}\lambda^2 t} \mid \mathcal{F}_s\right) = e^{\lambda X(s) - \frac{1}{2}\lambda^2 s}$$

$$M_\lambda(t) = e^{\lambda X(t) - \frac{1}{2}\lambda^2 t}$$ is

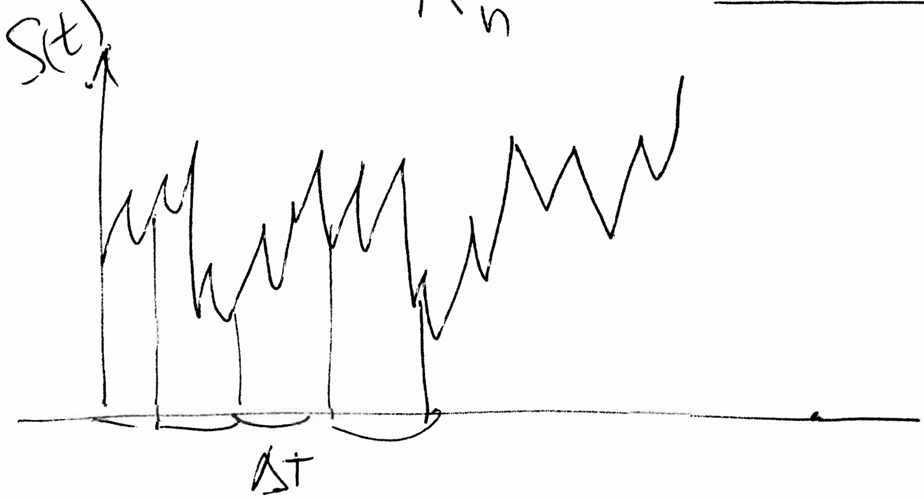
often called an <u>exponential martingale</u>

$$\boxed{E\left[M_\lambda(t) \mid \mathcal{F}_s\right] = M_\lambda(s)}$$

<u>Interpretation of $M_\lambda(t)$ as an econometric model.</u>

Suppose an economic variable – price returns – is assumed to be Gaussian with variance $\sigma^2 \Delta T$

$$R_n = \frac{S_{n\Delta t} - S_{(n-1)\Delta t}}{S_{(n-1)\Delta t}}$$



$$S_{n\Delta t} = S_0 \prod_{j=1}^{n} (1 + R_j)$$

constant

The idea of modeling $\sigma^2(R_n) = \sigma^2 \Delta T$ is so that the variance of returns matches a 1-year variance, for example. So, the assumption is that all intervals contribute equally to the variance (think CLT).

Calculate statistics of S_t

$$\ln \frac{S_{n\Delta t}}{S_0} = \sum_j \ln(1+R_j)$$

$$= \sum_j \ln(1+\sigma V_j \sqrt{\Delta T})$$

$$\simeq \sum_j \left[\sigma V_j \sqrt{\Delta T} - \frac{1}{2}\sigma^2 V_j^2 \Delta T\right]$$

If $t = n\Delta t$

$$S_t \simeq S_0 \, e^{\sum_{j=1}^{t/\sqrt{\Delta T}} \sigma V_j \sqrt{\Delta t} - \frac{1}{2}\sum_{1}^{t/\Delta t} \sigma^2 V_j^2 \Delta T}$$

$$\sum_{j=1}^{t/\sqrt{\Delta t}} \sigma V_j \sqrt{\Delta T} = \sigma \sum_{j=1}^{t/\sqrt{\Delta T}} V_j \sqrt{\Delta T}$$

$$= \sigma X(t) \quad (X = BM)$$

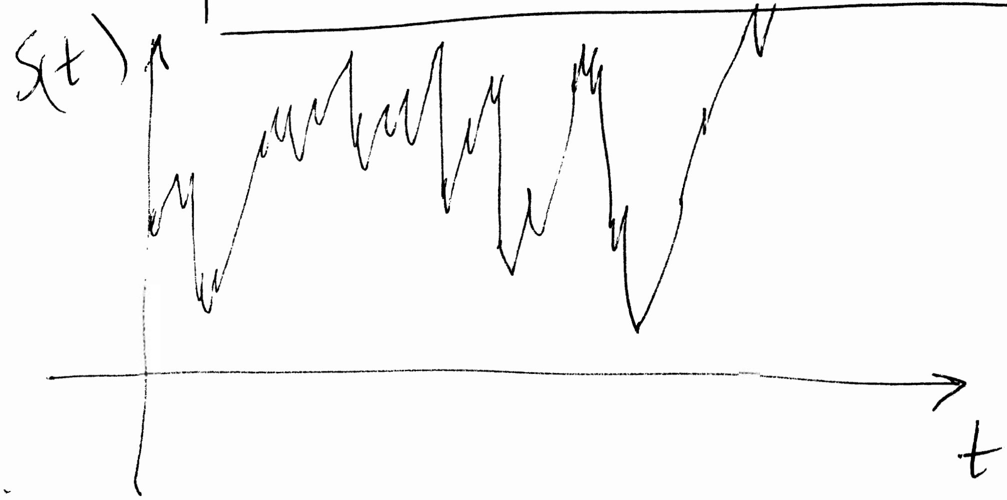
$$\sum_{1}^{t/\Delta t} \sigma^2 V_j^2 \Delta T = \sigma^2 \Delta T \sum V_j^2$$

$$= \sigma^2 t \left(\frac{\sum_1^n V_j^2}{n}\right)$$

$$\simeq \sigma^2 t \quad (n \to \infty)$$

$$M_{\sigma^2}(t) = S(t)$$

$$(**) \quad \boxed{S(t) = S(0)\, e^{\sigma \underline{X}(t) - \frac{1}{2}\sigma^2 t}}$$



This is sometimes called ~~an~~ ~~exponential~~ geometric Brownian motion because it is based on compounding rather than adding indept random variables.
Model (**) is used in the classical Black - Scholes - Merton option pricing model.
Another message: when you compound Gaussian r.v's the result is not the standard one. There is an extra term

in the exponential due to the variance.

Markov Property of Brownian motion

$$\boxed{\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}\{f(X_t) \mid X_s\}}$$

Proof: $f(X) = \int e^{ikX} \hat{f}(k)\, dk$

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = E\left[\int_{-\infty}^{+\infty} e^{ikX_t} \hat{f}(k)\, dk \mid \mathcal{F}_s\right]$$

$$= \int_{-\infty}^{+\infty} \hat{f}(k)\, dk\, E\left(e^{ikX_t} \mid \mathcal{F}_s\right)$$

$$= \int_{-\infty}^{+\infty} \hat{f}(k)\, dk\; e^{ikX_s}\, e^{-\frac{k^2}{2}(t-s)}\, dk$$

$=$ a function of X_s.

What function? This is easy to compute.

$$E\left[f(X_t) \mid X_s\right] =$$

$$\int_{-\infty}^{+\infty} f(X_s + y)\, e^{-\frac{y^2}{2(t-s)}} \frac{dy}{\sqrt{2\pi(t-s)}}$$

$$E\left[f(X_t) \mid X_s = x\right] = \int_{-\infty}^{+\infty} f(x+y)\, e^{-\frac{y^2}{2(t-s)}} \frac{dy}{\sqrt{2\pi(t-s)}}$$

$$= \int f(z)\, e^{-\frac{(x-z)^2}{2(t-s)}} \frac{dy}{\sqrt{2\pi(t-s)}}$$

$$= \psi(x, s)$$

$$\frac{\partial \psi}{\partial s} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} = 0 \,. \quad \psi(s=t, x) = f(x).$$

Since the Gaussian has a connection with heat diffusion, BM is connected with heat diffusion.