

## Stochastic Calculus I

This first assignment is a review of Probability in the form of 5 (somewhat lengthy) exercises. Make a special effort to understand the ideas used to solve the problems, because they will appear frequently in Stochastic Calculus. The main topics are: characteristic functions, conditioning and Gaussian distributions. Enjoy, M.A.

1. This exercise studies the characteristic function of random variables. We set  $\psi_X(k) = E(e^{ikX})$ .

(i) Let  $X, Y$  be independent random variables. Show that

$$\psi_{X+Y}(k) = \psi_X(k)\psi_Y(k).$$

(ii) Calculate the characteristic functions of the following distributions

- univariate gaussian  $(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

[Hint: Consider the case  $\mu = 0$ . The characteristic function of the Gaussian distribution is equal to

$$\psi(k) = \int_{-\infty}^{\infty} \cos(kx) e^{-\frac{x^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}}$$

You can compute the derivative of this function with respect to  $k$  and find that  $\psi'(k) = -\sigma^2 k \psi(k)$  from which the desired result follows.

- exponential

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

- Cauchy distribution

$$f(x) = \frac{1}{\pi} \frac{a}{a^2 + x^2}$$

- Poisson distribution

$$P\{X = k\} = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k \geq 0$$

• a sum of  $n$  independent exponentially distributed random variables  $X_1, X_2, \dots, X_n$  (with the same parameter  $\lambda$ ). (Hint: Use exercise 2). Verify that the characteristic function of

$$\frac{\sum_{k=1}^n X_k - n/\lambda}{\sqrt{n}}$$

converges to something nice as  $n \rightarrow \infty$ . Interpret this result...

2. A vector of random variables  $(X_1, \dots, X_m)$  is said to be *Gaussian* if, for any vector  $(\theta_1, \dots, \theta_m)$  of real numbers, the variable

$$Y = \sum_{j=1}^m \theta_j X_j$$

has a Gaussian distribution. Show that the general form of the ch.f. for a Gaussian random vector is

$$\psi(k_1, \dots, k_m) = \exp\left(i \sum_{j=1}^m \mu_j k_j - \frac{1}{2} \sum_{jj'=1}^m a_{jj'} k_j k_{j'}\right). \quad (1)$$

where  $\mu$  is a vector and  $a$  is a real, *non-negative definite matrix*. Characterize these quantities in terms of the moments of the random vector. (Hint: go directly to the Fourier side and write down what  $E\{exp^{ik \sum_i \theta_i X_i}\}$  should look like; equation (1) should follow).

3. The conditional expectation of a r.v.  $X$  with respect to another random variable  $Y$  is the *best nonlinear predictor* of  $X$  given  $Y$ , in the sense of least squares:

$$E(X|Y) = f(Y) \Leftrightarrow f(y) = \arg \min_{g(\cdot)} E(X - g(Y))^2 \quad (2)$$

(i) Deduce from the above definition that for any function  $\zeta(y)$

$$E(E(X|Y)\zeta(Y)) = E(X\zeta(Y))$$

which means that  $X - E(X|Y)$  is uncorrelated with any function of  $Y$ . (This, and equation (2) are in the spirit of “best predictor” of  $X$  given  $Y$ ).

(ii) Show that if the pair of random variables  $(X, Y)$  has joint probability density  $\phi(x, y)$  then

$$E(X|Y) = \frac{\int x\phi(x, y)dx}{\int \phi(x, y)dx}$$

(iii) Generalize the definition of conditional expectation to  $E(X|Y_1, \dots, Y_n)$ , where  $X, Y_1, \dots, Y_n$  are random variables.

(iv) The Tower Property holds:

$$E(E(X|Y, Z)|Z) = E(X|Z)$$

Generalize to conditioning to a vector of r.v.s. What should be the correct statement for conditioning with vectors?

(iv) A sequence  $X_1, X_2, \dots, X_n, \dots$  such that

$$E(X_n|X_{n-1}, X_{n-2}, \dots, X_1) = E(X_n|X_{n-1}) \quad (3)$$

is called a *martingale sequence*. Give an intuitive interpretation of (3) and an example of a martingale sequence.

(v) Suppose that you draw a number  $n$  in the interval  $(0, 1)$  at random (using the uniform distribution), and represent this number by its decimal expansion  $n = 0.a_1 a_2 a_3 a_4 \dots$ . Let

$$X_n = \sum_{i=1}^n a_i.$$

Is  $X_n$  a martingale? Why?

4. Show that if  $(X, Y)$  is a Gaussian vector, then  $E(X|Y) = \alpha + \beta Y$  for some constants  $\alpha, \beta$ . Give an example of a vector of random variables (non-Gaussian), for which this property does not hold.

5. Let  $X_n, n = 1, 2, 3, 4, \dots$  be a stochastic process defined by the iterative sequence

$$X_n = a + bX_{n-1} + \nu_n, n = 1, 2, 3, \dots$$

and assume that  $X_0$  is given, and that  $\nu_n$  is a sequence of independent Gaussian variables with variance 1 and mean zero. Express  $X_n$  as a function of  $X_0$  and  $\nu_1, \dots, \nu_n$  by solving the recursive relation inductively. Conclude that  $X_n$  is Gaussian for all  $n$  and that  $(X_1, \dots, X_n)$  is a Gaussian vector for all  $n$ . Show that if  $b < 1$  the distribution of  $X_n$  as  $n \rightarrow \infty$  is well-defined, is Gaussian, and give its mean and variance.