Stochastic Calculus

This first assignment is a review of Probability in the form of 5 (somewhat lengthy) exercises. Make a special effort to understand the ideas used to solve the problems, because they will *appear frequently* in Stochastic Calculus. The main topics are: characteristic functions, conditioning and Gaussian distributions. Enjoy, M.A.

1. This exercise studies the characteristic function of random variables. We set $\psi_X(k) = E(e^{ikX})$.

(i) Let X, Y be independent random variables. Show that

$$\psi_{X+Y}(k) = \psi_X(k)\psi_Y(k) .$$

(ii) Calculate the characteristic functions of the following distributions

• univariate gaussian (μ, σ^2)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

[Hint: Consider the case $\mu = 0$. The characteristic function of the Gaussian distribution is equal to

$$\psi(k) = \int_{-\infty}^{\infty} cos(kx) e^{\frac{-x^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}}$$

You can compute the derivative of this function with respect to k and find that $\psi'(k) = -\sigma^2 \, k \, \psi(k)$ from which the desired result follows.

• exponential

$$f(x) = \lambda e^{\lambda x} \quad , \ x > 0$$

• Cauchy distribution

$$f(x) = \frac{1}{\pi} \frac{a}{a^2 + x^2}$$

• Poisson distribution

$$P\{X = k\} = \frac{e^{-\lambda}\lambda^k}{k!} \ , \ k \geq 0$$

• a sum of n independent exponentially distributed random variables $X_1, X_2, ..., X_n$ (with the same parameter λ). (Hint:Use exercise 2). Verify that the characteristic function of

$$\frac{\sum_{k=1}^{n} X_k - n/\lambda}{\sqrt{n}}$$

converges to something nice as $n \to \infty$. Interpret this result...

2. A vector of random variables $(X_1, ..., X_m)$ is said to be *Gaussian* if, for any vector $(\theta_1, ..., \theta_m)$ of real numbers, the variable

$$Y = \sum_{j=1}^{m} \theta_j X_j$$

has a Gaussian distribution. Show that the general form of the ch.f. for a Gaussian random vector is

$$\psi(k_1, ..., k_m) = exp\left(i \sum_{j=1}^{m} \mu_j k_j - \frac{1}{2} \sum_{jj'=1}^{m} a_{jj'} k_j k_{j'} \right). \tag{1}$$

where μ is a vector and a is a real, *non-negative definite matrix*. Characterize these quantities in terms of the moments of the random vector. (Hint: go directly to the Fourier side and write down what $E\{exp^{ik \sum_i \theta_a X_a}\}$ should look like; equation (1) should follow).

3. The conditional expectation of a r.v. X with respect to another random variable Y is the *best nonlinear predictor* of X given Y, in the sense of least squares:

$$E(X|Y) = f(Y) \Leftrightarrow f(y) = \arg\min_{g(\cdot)} E\left(X - g(Y)\right)^2 \tag{2}$$

(i) Deduce from the above definition that for any function $\zeta(y)$

$$E\left(E(X|Y)\zeta(Y)\right) = E\left(X\zeta(Y)\right)$$

which means that $X - E(X|Y)$ is uncorrelated with any function of Y. (This, and equation (2) are in the spirit of "best predictor" of X given Y).

(ii) Show that if the pair of random variables (X, Y) has joint probability density $\phi(x, y)$ then

$$E(X|Y) = \frac{\int x\phi(x, y)dx}{\int \phi(x, y)dx}$$

(iii) Generalize the definition of conditional expectation to $E(X|Y_1, ..., Y_n)$, where $X, Y_1, ..., Y_n$ are random variables.

(iv) The Tower Property holds:

$$E(E(X|Y, Z)|Z) = E(X|Z)$$

Generalize to conditioning to a vector of r.v.s. What should be the correct statement for conditioning with vectors?

(iv) A sequence $X_1, X_2, ..., X_n, ...$ such that

$$E(X_n|X_{n-1}, X_{n-2}, ..., X_1) = E(X_n|X_{n-1}) \tag{3}$$

is called a *martingale sequence*. Give an intuitive interpretation of (3) and an example of a martingale sequence.

(v) Suppose that you draw a number n in the interval $(0, 1)$ at random (using the uniform distribution), and represent this number by its decimal expansion $n = 0.a_1\, a_2\, a_3\, a_4\, ...$ Let

$$X_n = \sum_{i=1}^{n} a_i.$$

Is X_n a martingale? Why?

4. Show that if (X, Y) is a Gaussian vector, then $E(X|Y) = \alpha + \beta Y$ for some constants α, β. Give an example of a vector of random variables (non-Gaussian), for which this property does not hold.

5. Let $X_n, n = 1, 2, 3, 4, ...$ be a stochastic process defined by the iterative sequence

$$X_n = a + bX_{n-1} + \nu_n \ , n = 1, 2, 3, ...$$

and assume that X_0 is given, and that ν_n is a sequence of independent Gaussian variables with variance 1 and mean zero. Express X_n as a function of X_0 and $\nu_1, ... \nu_n$ by solving the recursive relation inductively. Conclude that X_n is Gaussian for all n and that $(X_1, ..., X_n)$ is a Gaussian vector for all n. Show that if $b < 1$ the distribution of X_n as $n \to \infty$ is well-defined, is Gaussian, and give its mean and variance.