Recent Advances in Modeling Liquidity Risk and Applications to Central Clearing

Marco Avellaneda
New York University and Finance Concepts LLC
Outline

• Liquidity in risk management: listed markets, OTC Markets
• Liquidity-adjusted VaR for directional exposures
• BM&F Bovespa’s Close-out Risk Evaluation (CORE)
• The General Optimization Problem
• Solution of the Optimization Problem and Macro-Hedging
• Interest rate swaps
• Equity Options
Central clearing & risk-management

Large circles:
Clearing members (large banks, BDs)

Small circles:
non-clearing market participants (buy side)

Arrows represent credit exposure
Tools for Risk Management of CCPs

- Initial Margin (Market Risk, **Liquidity Risk**)
- Fund for mutualization of losses (``Guarantee Fund''), covering shortfall beyond the IM
- ``Loss tranches'' with CCP’s own capital (Skin in the game)

BIS, Principles for Financial Markets Infrastructures, BIS CPSS-IOSCO Consultative Report, April 2012

ESMA, Final Report, Technical Standards on OTC Derivatives, CCPs and Trade Repositories (``EMIR''), September 2012
Importance of Liquidity Risk Modeling

- LTCM (1998): basis trades (long high spreads/short low spreads, DV01=0)
- Credit Crunch (2007, 2008): loss of liquidity for financing structured paper
- J.P. Morgan CIO (2012): basis trade, index CDS versus corporate CDS (2bb loss -> 8 bb loss)
- MF Global (2012): 2y term repos on Italian government bonds
- Most Clearinghouses use Liquidity Charges as part of Margin Requirements for clearing members (IM and guarantee fund contributions)
Main Themes in Liquidity Risk

• Size & concentration: we must understand the market depth in a particular product and the relative liquidity between products

• Hedging: if you can hedge an illiquid instrument with a more liquid one, then the “package” is less risky. This means that it is easier to liquidate.

• Market structure matters: Exchanges are different than OTC venues (SEFs).

• Cost of liquidity: need to formulate liquidity as an optimization problem over some “cost function”

• Tractability: what good is an optimization problem if it can’t be solved?
Liquidating a defaulted CCP member

Initial jump triggers loss and default. Risk usually covered by Var/ES/CVaR ("market risk")

Large size implies further loss due to market impact ("liquidity risk")
Simple model assuming constant liquidation rate

\[ Q = \text{initial amount of contracts}, \; Q_0 = \text{average daily volume} \times \text{participation rate} \]

(Finish after \( N = \frac{Q}{Q_0} \) days)

PNL on day \( n \), based on ADV & participation rate:

\[ P(n) = \sum_{i=1}^{n} (Q - Q_0 i)\sigma \xi_i \]

Variance \( (P(N)) = \sum_{i=1}^{N} (Q - Q_0 i)^2 \sigma^2 \approx \frac{\sigma^2}{3} \frac{Q^3}{Q_0} \)
Constructing the Liquidity Charge

- Scaling gives a PNL for liquidation which is a standardized random variable multiplied by a nonlinear function of quantity.

- Any risk-measure (e.g. VaR, CVaR, STDEV) will give rise to a liquidation cost of the form

\[
\text{Cost} = \kappa \sigma Q_0 \left( \frac{Q}{Q_0} \right)^{1.5} = \kappa \sigma Q^{1.5} Q_0^{-0.5}
\]

- This is the asymptotics for large Q. Therefore, the Liquidity Charge should be something like

\[
L_C(Q) = \kappa \sigma Q \max \left( \left( \frac{Q}{Q_0} \right)^{0.5}, 1 \right)
\]

\[
\kappa \sigma \approx \frac{1}{2} \text{ (bid-ask spread) or } \text{``risk aversion'' coefficient (parameter to be calibrated)}
\]
Example: Eurodollar Futures (back of the envelope calculation)

- CME ED Futures: Median Volume ~ 100 K contracts
- Tick size (and bid/ask spread)= 0.25 bps for front month, 0.5 bps others.
- Contract sensitivity = USD 25 per LIBOR basis point move
- Reasonable volume = 10% daily volume = 10,000 contracts \( (Q_0) \)
- Typical liquid (large) trade = 10,000 \( \times \) 25 = 250,000 USD of DV01 \( (Q_0) \)
Liquidity Curves for ED futures

- Liquidity Charge for normal trades = $\frac{1}{2}$ bid-ask spread

Front month = 1/8 basis point
Out months = ¼ basis point

\[
LC(X) = \frac{1}{2} s \times \left(\frac{X}{X_0}\right)^{1.5}
\]

\[LC_f(X) = (0.125) \left(\frac{X}{0.25}\right)^{1.5} = X^{1.5}\]

\[LC_b(X) = (0.25) \left(\frac{X}{0.25}\right)^{1.5} = 2 X^{1.5}\]

<table>
<thead>
<tr>
<th>Spread</th>
<th>1 MM</th>
<th>5 MM</th>
<th>10 MM</th>
<th>25 MM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1</td>
<td>11</td>
<td>32</td>
<td>125</td>
</tr>
<tr>
<td>0.5</td>
<td>2</td>
<td>22</td>
<td>63</td>
<td>250</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>45</td>
<td>126</td>
<td>500</td>
</tr>
</tbody>
</table>

Liquidity charges in basis points of notional
For different sizes measured in MM Dv01
Liquidity Risk for Portfolios
Close-Out Risk Evaluation (CORE)  
Introduced by BM&F Bovespa, 2012

- Find a suitable liquidation strategy for each clearing member’s portfolio
- Compute potential uncollateralized losses associated with liquidation of the portfolio under stress scenarios
- Margin requirement is based on potential losses over the liquidation period

BM&FBOVESPA’s Post-Trade Infrastructure: Integration Opportunities and Challenges, September 2010,  
Modeling portfolios with liquidity constraints

- In a world with infinite liquidity, a portfolio is represented as a list of instruments and quantities.

<table>
<thead>
<tr>
<th>DOL Fut 01/2013</th>
<th>VALE5</th>
<th>GUAR3</th>
<th>BOVA11</th>
<th>IBOV Fut 04/2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,000</td>
<td>-45,000</td>
<td>53,000</td>
<td>-20,000</td>
<td>3,000</td>
</tr>
</tbody>
</table>

- In a world with limited liquidity, we should include the maximum amounts that can be traded in a given period (day) without ‘moving the market’.

<table>
<thead>
<tr>
<th>DOL Fut 01/2013</th>
<th>VALE5</th>
<th>GUAR3**</th>
<th>BOVA11</th>
<th>IBOV Fut 04/2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,000</td>
<td>-45,000</td>
<td>53,000</td>
<td>-20,000</td>
<td>3,000</td>
</tr>
<tr>
<td>25,000</td>
<td>1,000,000</td>
<td>1,000</td>
<td>150,000</td>
<td>10,000</td>
</tr>
</tbody>
</table>

* Proxied here at 10 % Avg. Traded Volume

** Guararapes Confecc. SA
Portfolio Description

<table>
<thead>
<tr>
<th>MTM(_1)(t,R)</th>
<th>MTM(_2)(t,R)</th>
<th>MTM(_3)(t,R)</th>
<th>MTM(_4)(t,R)</th>
<th>MTM(_5)(t,R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_1)</td>
<td>(Q_2)</td>
<td>(Q_3)</td>
<td>(Q_4)</td>
<td>(Q_5)</td>
</tr>
<tr>
<td>(l_1)</td>
<td>(l_2)</td>
<td>(l_3)</td>
<td>(l_4)</td>
<td>(l_5)</td>
</tr>
</tbody>
</table>

- \(R\) represents the state of the market or path of states of the market (risk-factor changes)
  
  \[
  R = (R_0, R_1, R_2, ... R_t, R_{t+1}, ...)
  \]

- Example: if we are dealing with options, then
  
  \[
  R_t = \begin{pmatrix}
  S_t \\
  \sigma_t \\
  r_t \\
  d_t
  \end{pmatrix}
  \]

- \(Q_i, l_i\) represent quantities and daily liquidity limits for each instrument
Liquidation of a Portfolio: `Close-out strategy’

- On date $t=0$, you decide that a portfolio should be liquidated starting on $t=1$.

- Determine a strategy in which a certain fraction, $q_{it}$, of the position in instrument $i$ will be liquidated at date $t$. ($q_{it}, i = 1, ..., N, t = 1, ..., T_{max}$)

\[
0 \leq q_{it} \leq \frac{l_i}{Q_i} \equiv k_i \quad \forall i \forall t
\]

\[
\sum_{t=1}^{T_{max}} q_{it} = 1
\]

\[
n_t = \sum_{s=t+1}^{T_{max}} q_s
\]

The remaining balance (%) at time $t$

- A close-out strategy is a matrix that tells us how to proceed for liquidating the various instruments in the portfolio as time passes.
Defining the objective function: P&L of a close-out strategy for a portfolio

\[ \psi_i(t, R_t) \overset{\text{def}}{=} Q_i[M_{TM_i}(t, R_t) - M_{TM_i}(0, R_0)] \]

- **Realized** P/L at date \( t \), after trading

\[ L_r(t, q, R_t) = \sum_{i=1}^{N} q_{it} \psi_i(t, R_t) \]

- **Unrealized** (a.k.a. MTM) P/L at date \( t \), after trading

\[ L_{nr}(t, q, R_t) = \sum_{i=1}^{N} n_{it} \psi_i(t, R_t) \]
Accumulated P/L

- Accumulated profit/Loss for close out strategy at date $t$

$$L(t, q, R) = \sum_{s=1}^{t} L_r(s, q, R_s) + L_{nr}(t, q, R_t)$$

$$= \sum_{s=1}^{t} \sum_{i=1}^{N} q_{is} \psi_i(s, R_s) + \sum_{i=1}^{N} n_{it} \psi_i(t, R_t)$$

- cash
- unrealized gain/loss
CORE objective function: The Worst Liquidity Shortfall under stress scenarios

- Define scenarios for the risk-factors, \( R = (R_1, R_2, \ldots, R_{T_{\text{max}}}) \)

- These scenarios are paths. Let \( R \) denote the set of all scenarios considered

\[
U(q) = \min_{R \in R} \min_{1 \leq t \leq T_{\text{max}}} L(t, q, R)
\]

\[
= \min_{R \in R} \min_{1 \leq t \leq T_{\text{max}}} \left( \sum_{s=1}^{t} \sum_{i=1}^{N} q_{is} \psi_i(s, R_s) + \sum_{i=1}^{N} n_{it} \psi_i(t, R_t) \right)
\]
The Optimization Problem

Maximize \( U(q) \) \( q = (q_{it}) \in \mathbb{R}^{N \times T_{\text{max}}} \)

Subject to:

\[
\begin{align*}
0 & \leq q_{it} \leq \frac{l_i}{Q_i} \equiv k_i \quad \forall i \ \forall t \\
\sum_{t=1}^{T_{\text{max}}} q_{it} & = 1 \quad \forall i
\end{align*}
\]

- \( U(q) \) is a sum of minima of linear functions of \( q \) \( \implies \) it is concave
- The set of constraints is convex (it is a convex polyhedral region)

A solution exists and should be unique under reasonable conditions!
Solution Via Linear Programming

Maximize: \( U \)

Variables: \( \{U, \lambda_t, \mu_t, q_{it}; 1 \leq t \leq T_{\text{max}}, 1 \leq i \leq N\} \)

Subject to constraints:

\[
\begin{align*}
U & \leq \lambda_t + \mu_t & \forall t \\
\lambda_t & \leq \sum_{i=1}^{N} q_{it} \psi_i(t, R_t) & \forall t \ \forall R \in R \\
\mu_t & \leq \sum_{i=1}^{N} (q_{it} + n_{it}) \psi_i(t, R_t) & \forall t \ \forall R \in R \\
0 & \leq q_{it} \leq \frac{l_i}{Q_i} \equiv k_i & \forall i \ \forall t \\
\sum_{t=1}^{T_{\text{max}}} q_{it} & = 1 & \forall i
\end{align*}
\]
Liquidity-adjusted Risk Margin

\[ M = \max \min_{q} \left( \min_{R \in R} \left( \min_{1 \leq t \leq T_{\text{max}}} L(t, q, R) \right) \right) \]

\[ = \min_{R \in R} \left( \min_{1 \leq t \leq T_{\text{max}}} L(t, q^*, R) \right) \]

\[ q^* = \text{optimal close-out strategy} \]

**Alternative versions**, which can be used with Monte Carlo models for RFs,

\[ \text{VaR}_\alpha \left( \min_{1 \leq t \leq T_{\text{max}}} L(t, q^*, R) \right) \quad \alpha = .99, \text{or} \ .995 \]

\[ \text{ES}_\alpha \left( \min_{1 \leq t \leq T_{\text{max}}} L(t, q^*, R) \right) \quad \alpha = .99 \ldots \]
Liquidation of a portfolio of stocks using the CORE risk-measure and Historical Monte Carlo constant liquidation speed gives rise to exponent $\sim 1.2-1.6$

- 10 stocks, 100 shares per stock

<table>
<thead>
<tr>
<th>SPY</th>
<th>GDX</th>
<th>UVXY</th>
<th>VTI</th>
<th>VWO</th>
<th>SIL</th>
<th>TLT</th>
<th>IWM</th>
<th>AGG</th>
<th>VOO</th>
</tr>
</thead>
</table>

Strategy $q$: liquidate equal amounts of stocks each day (not optimal)

$ES(X,T) = ES_{.99}\left(\min_{t \leq T} L(t, q, R)\right)$
General Optimization Problem

\[ Y_i(t) := MTM_i(R_t, t) \]  
Can assume that \( Y \) is a martingale

\[ dL(t) = \sum_{i=1}^{n} Q_i(t) dY_i(t) = Q(t) dY(t) \]

Optimization problem (mitigate expected shortfall with liquidity constraints)

\[
\max_{Q,T} ES_\alpha \left\{ \min_{t \leq T} \int_0^t Q(s) dY(s) \right\}
\]

Subject to \[ \left| \frac{dQ_i}{ds} \right| \leq k_i \quad i = 1, ..., n, \quad Q(0) \text{ given} \]
Solution of the General Optimization Problem

• In general, we are not aware of a closed-form solution for general models: use LP or QP

• For one asset ($n=1$) the optimal solution is to liquidate as fast as possible

\[ Q(t) = Q - kt, \quad T = \frac{Q}{k}. \]

• In the multivariate case, we conjecture, but cannot yet prove, that the solution should be a minimizer of the quadratic variation of $L(t)$, locally:

\[ \min_{Q(t)} \sum_{i,j=1}^{n} Q_i Q_j E(dL_i(t)dL_j(j)) \quad |\dot{Q}_i| \leq k_i \]

• One could, in principle, write some HJB equation for the general problem, but it does not seem to yield easily to a solution.
The Linearized (Gaussian) Case

\[ E\{dL_i dL_j\} = A_{ij} dt \quad \therefore \quad L(t) = \text{Brownian Motion} \]

\[ P(\min_{t \leq T} L(t) < a) = 2P(L(T) < a) \]

\[ ES_\alpha(\min_{t \leq T} L(t)) = ES_{\alpha'}(L(T)) = -z_{\alpha'} \sqrt{Var(L(T))} \]

\[ 1 - \alpha = 2(1 - \alpha') \quad \therefore \quad \alpha' = \frac{1 + \alpha}{2} \]

Program is:

\[
\begin{aligned}
\text{Minimize} \quad & \int_0^T \sum_{i,j=1}^n Q_i A_{ij} Q_j dt \\
\text{subject to} \quad & |\dot{Q}_i| \leq k_i
\end{aligned}
\]
Solution of the Linearized Problem

\[ \sum_{j=1}^{n} A_{ij} Q_j = 0 \text{ or } \dot{Q}_i = \pm k_i \]

Optimal liquidation strategy will maximize speed of execution or else lie in ``hyperplanes’’ in which the variance of the portfolio is minimized.

In practice, this is linked to the possibility of ``hedging’’ as part of the liquidation strategy.
2D problem

Asset 1: very liquid \( k_1 = \frac{1}{\varepsilon} \), \( \varepsilon \ll 1 \), 
Asset 2: normal \( k_2 = O(1) \)

\[ Q_1 = 0, Q_2 = \text{given} \]

Straight line is not optimal.
It is better to hedge first with
Liquid asset (e.g. futures)

\[ A_{11}Q_1 + A_{12}Q_2 = 0 \]
\[ Q_1 = -\frac{a \sigma_2}{\sigma_1} Q_2 = -\beta Q_2 \]
Optimal strategy: "macro hedge and unwind"

• Phase 1: take an offsetting position with liquid assets so as to minimize risk

\[
Q_1(t) = -k_1 t, \quad Q_2(t) = Q_2 - k_2 t \quad t \leq t^*
\]

• Phase 2: Slowly close out

\[
Q_1(t) = -\beta (Q_2 - k_2 t), \quad Q_2(t) = Q_2 - k_2 t \quad t > t^* \quad t^* = \frac{\beta Q_2}{k_1 + \beta k_2}
\]

• Admissibility condition \(|\beta k_2| \leq k_1\)

Phase 1 can be seen as a "hedging phase" which protects portfolio against drawdowns. Phase 2 is "slow liquidation" of hedged portfolio.
N-dimensional problem

- See Hongsik Kim’s NYU PhD thesis, from June 2014
- Qualitative behavior: (1) optimal trajectory hits a hyperplane
- (2) It remains in the hyperplane and then hits another hyperplane and continues on a codimension-2 linear subspace, etc.
- Each transition corresponds to putting in place an additional \`hedge\’.
- In practice, solutions are computed via QP.
- Strong overlap with Bovespa CORE. Also seen in Almgren-Chriss execution model under separation of scales.
Modelling the Liquidation Cost in Practice

In the limit $\varepsilon \ll 1$, assuming $N$ moderately liquid assets and one hedging asset

Liquidity Cost = (Cost of Macro-Hedge) + (Cost of Liquidating Hedged Portfolio)

\[
LC = \alpha_1 \left| \sum_{i=1}^{N} \beta_i Q_i \right|^{1.5} + \alpha_2 \sum_{i=1}^{N} |Q_i| \max \left( \sqrt{\frac{|Q_i|}{k_i}}, 1 \right)
\]

The constants $\alpha_i$ should be fitted to data on liquidation costs or to polls (depends on asset class, market structure.)
Liquidity in OTC Risk-Management

- In OTC, the portfolio is liquidated in an auction (there is no exchange).

- Participants periodically inform the CCP on liquidity and market depth, so IM requirements can take into account liquidation costs.

- The time dimension of liquidation should be “made equivalent” to a wider B/O spread in a 1-day auction.

- The second element in the LC corresponds to the estimated transaction cost in auctioning the “macro-hedged” portfolio.
Polling the Clearing Members

- Polls are conducted asking CCP participants by how much would their bid or ask price would change as a function of trade size.

- Liquidity polls typically involve
  - directional (outright) positions
  - market-neutral portfolios

- Liquidity charges for market-neutral portfolios are typically lower than for outright positions because they have less risk exposure.

- To some extent, the poll incorporates the time-dimension of the close-out process.
Example: IR Swaps

We consider 4 standard swap tenors. A typical poll will consider several portfolios: outright swap positions, curve positions (or time spreads) and butterfly spreads. Portfolios 5 to 13 are market-neutral.

<table>
<thead>
<tr>
<th>Tenor (yrs)</th>
<th>Swap positions (in MM DV01)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Port 1</td>
<td>1</td>
</tr>
<tr>
<td>Port 2</td>
<td>1</td>
</tr>
<tr>
<td>Port 3</td>
<td></td>
</tr>
<tr>
<td>Port 4</td>
<td></td>
</tr>
<tr>
<td>Port 5</td>
<td>1</td>
</tr>
<tr>
<td>Port 6</td>
<td>1</td>
</tr>
<tr>
<td>Port 7</td>
<td>1</td>
</tr>
<tr>
<td>Port 8</td>
<td>1</td>
</tr>
<tr>
<td>Port 9</td>
<td></td>
</tr>
<tr>
<td>Port 10</td>
<td>1</td>
</tr>
<tr>
<td>Port 11</td>
<td></td>
</tr>
<tr>
<td>Port 12</td>
<td>1</td>
</tr>
<tr>
<td>Port 13</td>
<td>1</td>
</tr>
</tbody>
</table>
Liquidity Charge Curves obtained by polling 10 dealers and taking median values

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>multiplier (1X, 5X, 10X, 25X)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1.25</td>
</tr>
<tr>
<td>2</td>
<td>1.93</td>
</tr>
<tr>
<td>3</td>
<td>2.25</td>
</tr>
<tr>
<td>4</td>
<td>3.00</td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
</tr>
<tr>
<td>6</td>
<td>2.00</td>
</tr>
<tr>
<td>7</td>
<td>2.00</td>
</tr>
<tr>
<td>8</td>
<td>2.00</td>
</tr>
<tr>
<td>9</td>
<td>2.00</td>
</tr>
<tr>
<td>10</td>
<td>2.00</td>
</tr>
<tr>
<td>11</td>
<td>2.00</td>
</tr>
<tr>
<td>12</td>
<td>4.00</td>
</tr>
<tr>
<td>13</td>
<td>4.00</td>
</tr>
</tbody>
</table>

- We use the median value as an indicator of the function $F(N)$ for each spread
- Rows 1 and 2 are similar to what we obtained earlier for ED futures based on 1.5 model
Smoothing the Poll Data

- Take the discrete poll and fit the data to power-laws using log-log regression

Outright 2Y swap

\[ F(N) = (1.26827) \times N^{1.6406} \]

Exponent greater Than 1

Bps for 1MM DV01
Empirical results match 1.5 model

Liquidity Charge Curves (in bps per USD 1 million DV01)

<table>
<thead>
<tr>
<th>Spread</th>
<th>ln a</th>
<th>a</th>
<th>b</th>
<th>R-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.23765327</td>
<td>1.268269369</td>
<td>1.640640912</td>
<td>0.990944737</td>
</tr>
<tr>
<td>2</td>
<td>0.638018832</td>
<td>1.892727352</td>
<td>1.524495236</td>
<td>0.979830862</td>
</tr>
<tr>
<td>3</td>
<td>0.7800459</td>
<td>2.181572398</td>
<td>1.558456377</td>
<td>0.993419153</td>
</tr>
<tr>
<td>4</td>
<td>1.030187386</td>
<td>2.801590763</td>
<td>1.555690894</td>
<td>0.972527236</td>
</tr>
<tr>
<td>5</td>
<td>7.46475E-05</td>
<td>1.00007465</td>
<td>1.607219352</td>
<td>0.983037657</td>
</tr>
<tr>
<td>6</td>
<td>0.65997447</td>
<td>1.93474294</td>
<td>1.45941347</td>
<td>0.981681918</td>
</tr>
<tr>
<td>7</td>
<td>0.667487666</td>
<td>1.949333786</td>
<td>1.549304728</td>
<td>0.958037632</td>
</tr>
<tr>
<td>8</td>
<td>0.608367126</td>
<td>1.837428659</td>
<td>1.411071285</td>
<td>0.956494481</td>
</tr>
<tr>
<td>9</td>
<td>0.652015481</td>
<td>1.919405458</td>
<td>1.501764209</td>
<td>0.990402548</td>
</tr>
<tr>
<td>10</td>
<td>0.640578078</td>
<td>1.897577509</td>
<td>1.571858782</td>
<td>0.951821306</td>
</tr>
<tr>
<td>11</td>
<td>0.717869273</td>
<td>2.050060434</td>
<td>1.616243426</td>
<td>0.989358386</td>
</tr>
<tr>
<td>12</td>
<td>1.342315061</td>
<td>3.827895065</td>
<td>1.548779256</td>
<td>0.966291357</td>
</tr>
<tr>
<td>13</td>
<td>1.356325854</td>
<td>3.882009196</td>
<td>1.511079491</td>
<td>0.984376</td>
</tr>
</tbody>
</table>

Table 1: Results of log-log regression for the 13 spreads in the swaps example, using median poll values. The fit is to the curve $F = aN^b$ which is done by estimating the linear model $lnF = lna + b(lnN) + \epsilon$. The $R^2$ coefficient is given in the right-most column. Notice that the empirically estimated exponents $b$ are reasonably close to the $b=1.5$ heuristic value.
Liquidity Add-on Charge Calculation for IR Swap Portfolios

- Represent swap portfolios as loadings on the standard tenor swaps (2y, 5y, 10y, 30y)

- Minimize cost of liquidation:

  \[ \sum_{i=1}^{N} F_i(Q_i) \]

  subject to the linear constraints

  \[ \sum_{i=1}^{N} \mu_{im} Q_i = \Delta_m \quad m = 1, \ldots, M \]

  and bound constraints

  \[ |Q_i| \leq Q_{max,i} \quad i = 1, \ldots, N \]
Example

**TARGET PORTFOLIO**

<table>
<thead>
<tr>
<th>TENOR</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>DV01 MM</td>
<td>12</td>
<td>-18</td>
<td>5</td>
<td>-5</td>
</tr>
<tr>
<td>CHARGE (bp)</td>
<td>74.78</td>
<td>155.15</td>
<td>22.01</td>
<td>34.26</td>
</tr>
<tr>
<td>NAÏVE CHG</td>
<td>286.20</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**CALCULATED SPREAD**

<table>
<thead>
<tr>
<th>SPREAD</th>
<th>HEDGE POSITION</th>
<th>LIQUIDITY</th>
<th>CHARGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>-0.46</td>
<td>0.59</td>
<td>0.38</td>
</tr>
<tr>
<td>3</td>
<td>0.33</td>
<td>0.38</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>-5.86</td>
<td>43.86</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>17.72</td>
<td>101.57</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td>-4.64</td>
<td>18.14</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>-1.09</td>
<td>2.22</td>
<td>0.00</td>
</tr>
<tr>
<td>8</td>
<td>0.20</td>
<td>0.19</td>
<td>0.00</td>
</tr>
<tr>
<td>9</td>
<td>0.22</td>
<td>0.20</td>
<td>0.00</td>
</tr>
<tr>
<td>10</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>11</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Sum of changes</td>
<td>167.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NAÏVE CHARGE</td>
<td>286.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SMART CHARGE</td>
<td>167.15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Exp Shortfall 99% 234,136,074 (Margin)

Liquidity Charge 167,145,075
Exchange-traded equity options

- Listed option portfolios in the US markets can have 1000+ underlying assets with 100s of strikes.

- Open interest and daily traded volume is very large in aggregate, but not necessarily so for a given asset/tenor/strike.

- Bid-ask spreads can be very large compared to premium (e.g. 50 cents on a 5-dollar option).

- It is possible to use the method developed above to construct liquidity charges that can be used by a Central Counterparty (CCP) to margin its participants.
Liquidity Charge for Option Portfolios

\[ LC = LC_1 + LC_2 + LC_3 + LC_4 \]

\( LC_1 \): Macro Delta-hedge. Instruments: S&P 500 futures (front month)

\( LC_2 \): Macro Vega-hedge. Instruments: ATM S&P 500 options (short, middle, long, tenors)

\( LC_3 \): Cost of auctioning a delta-neutral / vega-neutral portfolio (post macro hedge)

\( LC_4 \): Cost of replacing the S&P futures delta-hedge by actual hedged (Delta basis risk)

In each case, we use the 1.5 template with ADV for determining liquidation speed. Pre-factors \( (\alpha_i) \) are calibrated using bid-ask spreads, polls and dealer estimates.
Independent Validation: comparing with the LC of an anonymous large dealer

The LC developed here was calibrated and then compared with the LC of a large dealer. The agreement is excellent at the aggregate level, but there are differences from one account to the next.
Conclusions

• Liquidity Modeling is an integral part of risk management.

• Models typically include portfolio size, market depth, and time-horizon to estimate cost.

• CORE (BMF&F Bovespa) suggests constructing liquidity thresholds for each security and finding a liquidation strategy that mitigates close-out cost.

• Linearization of the CORE approach gives rise to a Quadratic Optimization problem which resembles other frameworks for portfolio execution with impact cost. Liquidation time is endogenous.

• Solutions are easy to describe in terms of macro-hedging / unwinding hedged portfolios.

• This approach is used to build LCs in practice for CCPs. The charges are based on a 1.5 – total cost model, with ADV and bid-ask spreads used as inputs.

• Additional parameters must be introduced (cost for hedged portfolios). This is done by polling dealers and other market participants and checking results via blinded tests.