Derivative Securities: Lecture 4

Option Pricing: Arrow-Debreu approach
A general approach to derivative pricing

• Assume a 1-period model: trades are done at date $t=0$ and the payoff date is $t=T$.

• Let $C(K, T)$ represent the fair value of a call option with strike $K$ maturing on date $T$.

• Our first goal is to find a suitable pricing formula from first principles.

• Difficulty: we do not know the final state of the stock and the static cash and carry argument that we used for forwards does not apply. (Why?)
Butterfly spreads

- A butterfly spread is a position in options which corresponds to:
  - long 1 call with strike K-a
  - short 2 calls with strike K
  - long 1 call with strike K+a

Payoff diagram for butterfly spread
Butterfly spreads and call prices

• Since the payoff of a butterfly is non-negative in all future states, its value should be positive

\[ B(K - a, K, K + a, T) = C(K - a, T) - 2C(K, T) + C(K + a, T) \]

• Consider \( 1/a^2 \) butterfly spreads in the limit \( a \to 0 \):

\[
\lim_{a \to 0} \frac{B(K - a, K, K + a, T)}{a^2} = \lim_{a \to 0} \frac{C(K - a, T) - 2C(K, T) + C(K + a, T)}{a^2}
\]

\[
= \frac{\partial^2 C(K, T)}{\partial K^2} = B(K, T) \geq 0
\]

• Conclusion: Call values should be convex in \( K \) (and decreasing).
Arrow-Debreu prices

We have

\[ \int_0^\infty B(X,T)dX = e^{-rT} \]

Proof: for \( K \ll 1 \), \( C(K,T) \approx PV(F - K) \);

for \( K \gg 1 \), \( C(K,T) \approx 0 \)

\[ \frac{\partial C(K,T)}{\partial K} \rightarrow -e^{-rT} \text{ as } K \rightarrow 0 \]

\[ \frac{\partial C(K,T)}{\partial K} \rightarrow 0 \text{ as } K \rightarrow \infty \]

\[ \int_0^\infty B(X,T)dX = \left[ \frac{\partial C(K,T)}{\partial K} \right]_{K=0,\infty} \]
AD Probabilities and Option Prices

\[ p(X, T) = e^{rT} B(K, T) \]  

Arrow-Debreu Probabilities

\[ \int_0^\infty p(X, T) dX = 1 \]

**Proposition 1:** Expected value under AD= forward price

\[ \int_0^\infty X p(X, T) dX = F_T \]

**Proposition 2:** The fair values of puts and calls can be represented as their expectations under the AD probability

\[ C(K, T) = e^{-rT} \int_0^\infty \max(X - K, 0) p(X, T) dX \]

\[ P(K, T) = e^{-rT} \int_0^\infty \max(K - X, 0) p(X, T) dX \]
Proof of Proposition 1

Proposition 1 follows from Proposition 2 because, from PCP,

\[ C(K,T) - P(K,T) = e^{-rT}(F_T - K). \]

\[
C(K,T) - P(K,T) = e^{-rT} \int_0^\infty \max(X - K, 0)p(X,T)dX -
\]

\[
e^{-rT} \int_0^\infty \max(K - X, 0)p(X,T)dX
\]

\[
= e^{-rT} \int_0^\infty (X - K)p(X,T)dX
\]

\[
= e^{-rT} \int_0^\infty Xp(X,T)dX - e^{-rT}K
\]

\[ \therefore \int_0^\infty Xp(X,T)dX = F_T \]
Proof of Proposition 2

$$\frac{\partial}{\partial K} e^{-rT} \int_0^\infty \max(X - K, 0)p(X,T)dX =$$

$$= - e^{-rT} \int_0^\infty H(X - K)p(X,T)dX$$

$$= - e^{-rT} \int_K^\infty p(X,T)dX$$

$$\frac{\partial^2}{\partial K^2} = e^{-rT} p(K,T) = B(K,T) = \frac{\partial^2 C(K,T)}{\partial K^2}$$

• Thus, the integral expression and the call price $C(K,T)$ differ at most by a linear function of $K$.

• It is trivial to show that since $C(0,T) = PV(F_T)$, $C(\infty, T) = 0$, the linear function is zero. The proof is the same for puts.
Black Scholes model

• Assume that AD measure is log-normal. In other words, the AD probability is the distribution of a random variable

\[ X = F_T e^{aZ+b} \]

where \( Z \) is normal N(0,1) and \( a \) and \( b \) are parameters. Also we need from Prop 1, that

\[ 1 = E(e^{aZ+b}) = e^{\frac{a^2}{2}+b} \]

which implies \( b = -\frac{a^2}{2} \), or \( X = F_T e^{aZ-\frac{a^2}{2}} \).

• The parameter \( a \), which is not a financial parameter (as the forward) corresponds to the standard deviation of log-returns, over the time horizon.
Volatility

- The standard deviation of log returns is called volatility (option volatility, implied volatility)

- The units of volatility for equity derivatives and for FX (but not for all IR derivatives is % change per year.

- If we denote the volatility by $\sigma$ and measure time in years, we have

$$a = \sigma \sqrt{T}$$

and the BS model for the AD probabilities reads

$$X = F_T e^{\sigma \sqrt{T} Z - \frac{1}{2} \sigma^2 T}$$

- This means that $p(X, T)$ is the density of this random variable.
The Black-Scholes Formula

Plugging in the density for the log-normal random variable, we find that

\[ C(K, T) = e^{-rT} (F_T N(d_1) - KN(d_2)) \]

where \( N(x) \) is the cumulative distribution for a standardized normal r.v. and

\[ d_1 = \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{F_T}{K} \right) + \frac{\sigma^2 T}{2} \]

\[ d_2 = \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{F_T}{K} \right) - \frac{\sigma^2 T}{2} \]

This formula is used millions of times every day in exchanges worldwide.
The BS Formula with Spot quantities

- The BS formula also can be written in terms of `spot’’ quantities

\[ \text{BSCall}(S, T, K, r, q, \sigma) = e^{-qT}SN(d_1) - e^{-rT}KN(d_2) \]

\[ d_1 = \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{S}{K} \right) + \frac{(r-q)\sqrt{T}}{\sigma} + \frac{\sigma^2 T}{2} \]

\[ d_2 = \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{S}{K} \right) + \frac{(r-q)\sqrt{T}}{\sigma} - \frac{\sigma^2 T}{2} \]

- It shows that the value of an option depends not only on the cost-of-carry but also on the volatility parameter.

- For historical reasons, the above formula is known as the Black-Scholes formula, whereas the one with the forward is known as Black’s formula. Of course, they express exactly the same idea.