A Ramsey-type Result for Convex Sets

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Abstract

Given a family of n convex compact sets in the plane, one can always choose $n^{1/5}$ of them which are either pairwise disjoint or pairwise intersecting. On the other hand, there exists a family of n segments in the plane such that the maximum size of a subfamily with pairwise disjoint or pairwise intersecting elements is $n^{\log 2/\log 5} \leq n^{0.431}$.

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1 Introduction

Ramsey's theorem [11], [7] states that any graph of n vertices contains either a complete or an empty subgraph of $\frac{1}{2}\log n$ vertices. Erdös [5] showed that the order of magnitude of this bound cannot be improved. However, much stronger results are expected for some special classes of graphs, e.g. for graphs representing the intersection pattern of some family of geometric figures.

The best known example of this kind is the following. Given any family $\mathcal{I} = \{I_1, \ ..., \ I_n\}$ of intervals on a line with the property that no point is contained in more than p members of $\mathcal{I}$, one can always decompose $\mathcal{I}$ into at most p disjoint subfamilies so that each of them consists of pairwise disjoint intervals (see e.g. [3], [10]). This immediately imp the graph defined on the vertex set $\mathcal{I}$ by joining I_i and I_j with an edge if and only if $I_i \cap I_j \neq 0$, contains either a complete or empty subgraph of size $\lceil \sqrt{n} \ \rceil$. Pach [8] has pointed out that this result can be extended to any family $\mathcal{C} = \{C_1, \ ..., \ C_n\}$ of convex bodies in $\mathcal{R}^d$ with the property that no C_i is too 'longish', i.e., the ratio between the circumradius and the inradius of C_i is smaller than some constant.

The first interesting open problem that arises is the following: Determine (or estimate) the largest number $g(n)$ with the property that any family of n segments in the plane in general position contains at least $g(n)$ members that are either pairwise disjoint or pairwise crossing [9] (see also [1], p.19). Segments in the plane are of course infinitely 'longish' in the above sense.

In the present paper we shall address this question in a more general setting. Our main result is the following.

Theorem 1 *Let $f(n)$ denote the maximum number with the property that given a family of*

n convex compact sets in the plane, one can always choose $f(n)$ of them which are either pairwise disjoint or pairwise intersecting. Then

$$n^{0.2} \leq f(n) \leq n^{\log 2/\log 5} \leq n^{0.431}$$

Let us remark that we cannot expect any superlogarithmic lower bound to hold for the analogously defined functions in higher dimensions. This follows from a result of Tietze [12] (rediscovered by Besicovitch [2]), which shows that any graph can be represented as the intersection pattern of a family of 3-dimensional convex bodies.

If instead of general convex compact sets in the plane, we consider rectangles with sides parallel to the x and y axes, then we can prove the following lower bound.

Theorem 2 *Let $r(n)$ denote the maximum number with the property that given a family of n rectangles in the plane with sides parallel to the x and y axes, one can always choose $r(n)$ of them which are either pairwise disjoint or pairwise intersecting. Then*

$$\sqrt{\frac{n}{2\log_2 n}} \leq r(n).$$

Notation. Throughout this paper, let log stand for logarithm to base 2. For a finite family $\mathcal{F}$ of sets, let $D(\mathcal{F})$ denote the maximum size of a pairwise disjoint subfamily of $\mathcal{F}$, and $I(\mathcal{F})$ the maximum size of a pairwise intersecting subfamily of $\mathcal{F}$.

2 Proof of Theorem 1

We need the following well-known result of Dilworth [4] (actually the statement which we need is much easier to prove than the full version of Dilworth's theorem).

Lemma 2.1 *For any positive integers n, p, every partial order on at least n elements contains either a chain (totally ordered subset) of length p or an antichain (a subset of pairwise incomparable elements) of size $\lceil \frac{n}{p} \rceil$.*

Let $\mathcal{C}$ be a family of n compact convex sets. For $C \in \mathcal{C}$, let $\pi(C)$ denote the projection of C onto the x-axis. We define four binary relations $\prec_1$, $\prec_2$, $\prec_3$ and $\prec_4$ on $\mathcal{C}$. One necessary condition for $A, B \in \mathcal{C}$ to be in any of these relations is $A \cap B = \emptyset$. For disjoint $A, B \in \mathcal{C}$, we define (see Fig. 1)

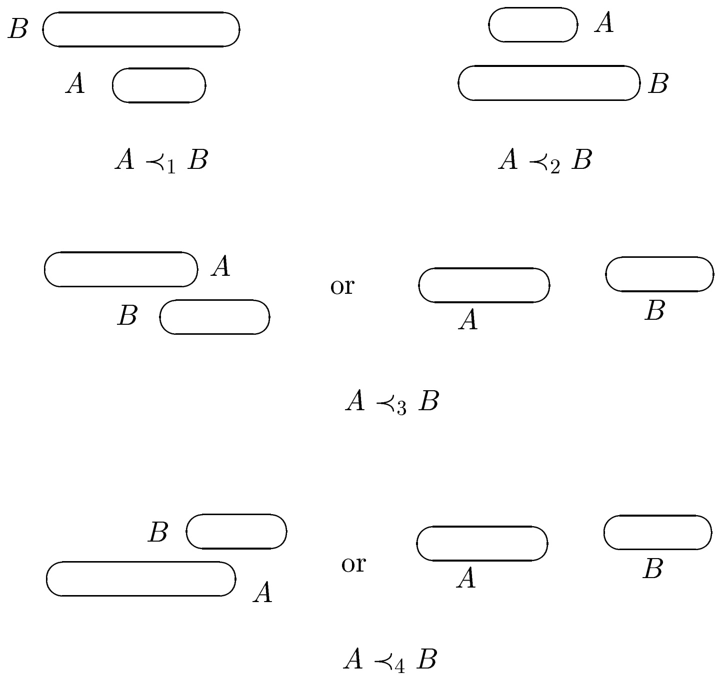

Figure 1: The relations $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$

- $A \prec_1 B$ if $\pi(A) \subseteq \pi(B)$ and A lies below B ("below" means in the y-axis direction).

- $A \prec_2 B$ if $\pi(A) \subseteq \pi(B)$ and A lies above B.

- $A \prec_3 B$ if the left endpoint of $\pi(B)$ is to the right of the left endpoint of $\pi(A)$, the right endpoint of $\pi(B)$ is to the right of the right endpoint of $\pi(A)$ and in the part where $\pi(A)$ and $\pi(B)$ overlap (if any), A lies above B.

- $A \prec_4 B$ if the left endpoint of $\pi(B)$ is to the right of the left endpoint of $\pi(A)$, the right endpoint of $\pi(B)$ is to the right of the right endpoint of $\pi(A)$ and in the part where $\pi(A)$ and $\pi(B)$ overlap (if any), A lies below B.

We need the following elementary claims, whose verification is left to the reader:

Lemma 2.2 *Each of the relations $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$ is transitive.* $\square$

Lemma 2.3 *For any two disjoint convex sets $A, B \in \mathcal{C}$, $A \prec_i B$ for some i or $B \prec_i A$ for some i.* $\square$

Since a necessary condition on a pair of sets belonging to any of the relations $\prec_i$ is that they are disjoint, a any two sets in a chain are disjoint. Let us first look at the relation $\prec_1$ on $\mathcal{C}$. By Dilworth's theorem 2.1, either there is a chain of $n^{1/5}$ elements in $\prec_1$, or an antichain $\mathcal{A}_1$ of at least $n^{4/5}$ elements. In the first case we have $n^{1/5}$ pairwise disjoint sets in $\mathcal{C}$ and we are done, so let us assume the latter case. We look at $\prec_2$ restricted to $\mathcal{A}_1$. Either there is a chain of $n^{1/5}$ elements (in which case we are done) or an antichain $\mathcal{A}_2$ of at least $n^{3/5}$ elements (of $\mathcal{A}_1$). Looking now at $\prec_3$ restricted to $\mathcal{A}_2$, we either obtain a chain with $n^{1/5}$ elements or an antichain $\mathcal{A}_3$ with $n^{2/5}$ elements. Finally considering $\prec_4$ restricted to $\mathcal{A}_3$ either gives an $n^{1/5}$ element chain or an $n^{1/5}$ element antichain $\mathcal{A}_4$. Such $\mathcal{A}_4$ is an antichain in each of $\prec_1, \ldots, \prec_4$, and so by Lemma 2.3 any two sets from $\mathcal{A}_4$ must intersect. This proves the lower bound in Theorem 1.

To establish the upper bound, we need the following.

Construction 2.4 *For any two points p, q and any $\varepsilon > 0$, there exist five segments p_1q_1, p_2q_2, ..., p_5q_5 such that p_iq_i intersects only the segments $p_{i-1}q_{i-1}$ and $p_{i+1}q_{i+1}$, and all p_i (all q_i) are inside the circle around p (q) with radius ε (the indices are taken mod 5. See Fig. 2.)*

Figure 2: A representation of the 5-cycle by segments

Let $\mathcal{C}_1$ be such a family of five segments. Given $\mathcal{C}_i$, choose ε to be so small that no disk of radius ε around an endpoint of a segment of $\mathcal{C}_i$ meets any other segment of $\mathcal{C}_i$. To obtain $\mathcal{C}_{i+1}$, replace each segment pq of $\mathcal{C}_i$ by a family of five segments satisfying the conditions of the construction.

Obviously, $|\mathcal{C}_i| = 5^i$ and the maximum size of a subfamily of $\mathcal{C}_i$ with pairwise disjoint or pairwise intersecting elements is $2^i = |\mathcal{C}_i|^{\log 2/\log 5} \leq |\mathcal{C}_i|^{0.431}$ for every i.

3 Proof of Theorem 2

For an integer m, let $\psi(m)$ denote the minimum n such that for any family $\mathcal{R}$ of n axis-parallel rectangles the product $D(\mathcal{R})I(\mathcal{R})$ is at least m. We prove that $\psi(2^k) \leq k2^k$, for $k = 1, 2, \ldots$. This is enough to prove Theorem 2: Given a family $\mathcal{R}$ of n axis-parallel rectangles, set $k = \lfloor \log n - \log\log n \rfloor$. Then $\psi(2^k) \leq k2^k \leq n$, so $D(\mathcal{R})I(\mathcal{R}) \geq 2^k \geq 2^{\log n - \log\log n - 1} = \frac{n}{2\log n}$, and Theorem 2 follows.

We proceed by induction on k. We have $\psi(2) = 2$, verifying the claim for $k = 1$. Let $k > 1$ and consider a family $\mathcal{R}$ of $k2^k$ axis-parallel rectangles. It is easy to see that we may assume that all the lines defined by the right vertical sides of the rectangles of $\mathcal{R}$ are distinct. Let v be the tth leftmost of these vertical lines, where $t = (k-1)2^{k-1}$. Let $\mathcal{R}_0 \subseteq \mathcal{R}$ be the subfamily of rectangles intersecting v. We distinguish two cases. First, if $|\mathcal{R}_0| \geq 2^k$, we have $D(\mathcal{R})I(\mathcal{R}) \geq D(\mathcal{R}_0)I(\mathcal{R}_0) \geq 2^k$, by the result concerning intervals mentioned in the introduction. Second, suppose that $|\mathcal{R}_0| \leq 2^k$, and let $\mathcal{R}_l$ be the family of rectangles of $\mathcal{R}$ lying in the left closed halfplane defined by v and $\mathcal{R}_r$ the family of rectangles of $\mathcal{R}$ lying strictly to the right of v. We have $|\mathcal{R}_l| = (k-1)2^{k-1}$ by the choice of v, and $|\mathcal{R}_r| \geq |\mathcal{R}| - |\mathcal{R}_0| - |\mathcal{R}_l| \geq (k-1)2^{k-1}$. By the inductive hypothesis, we have $D(\mathcal{R}_l)I(\mathcal{R}_l) \geq 2^{k-1}$, $D(\mathcal{R}_r)I(\mathcal{R}_r) \geq 2^{k-1}$. Then

$$D(\mathcal{R})I(\mathcal{R}) \geq [I(\mathcal{R}_l) + I(\mathcal{R}_r)] \max[D(\mathcal{R}_l), D(\mathcal{R}_r)] \geq I(\mathcal{R}_l)D(\mathcal{R}_l) + I(\mathcal{R}_r)D(\mathcal{R}_r) \geq 2^k\,,$$

which finishes the inductive proof. □

Let us remark that for n of the form $k2^k$, we have proved the bound in Theorem 2 with constant 1. We could make this improvement in general, by careful calculations with various lower and upper fractional parts, but this would bring technical complications into our simple proof.

4 Remarks

1. Theorem 1 also holds for any family of compact connected sets that are convex in one direction, i.e., any line parallel to this direction intersects each member of $\mathcal{C}$ in a single segment.

2. The graphs used in the proof of the upper bound of Theorem 1 can also be represented as intersection graphs of (left-infinite) halflines (it is easy to see that the segments in Fig. 2 can be extended to left-infinite halflines). Thus, the $n^{0.431}$ upper bound is also valid for intersection graphs of halflines. On the other hand, the lower bound proof in Theorem 1 can be modified to get $n^{1/3}$ lower bound for families of halflines (since any two disjoint halflines are comparable in one of the relations $\prec_1$, $\prec_2$).

3. The proof method of Theorem 2 can be used in various other situations; essentially it shows that we can concentrate on the situation when all sets of the considered family intersect a common vertical line. For instance, it gives the following result:

   Lemma 4.1 *Let $\mathcal{F}$ be a family of n sets in the plane, such that for every subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ intersecting a common vertical line we have $I(\mathcal{F}_0)D(\mathcal{F}_0) \geq Cm^\alpha$, where $m = |\mathcal{F}_0|$ and C, α are constants, $0 < \alpha \leq 1$. Then*

$$I(\mathcal{F})D(\mathcal{F}) \geq \begin{cases} C'n^\alpha & \text{for } \alpha < 1 \\ \frac{Cn}{2\log n} & \text{for } \alpha = 1 \end{cases}$$

   (C' is a constant depending on C, α). □

4. It seems likely that the bound in Theorem 2 can be improved to $r(n) \geq c\sqrt{n}$ for a suitable constant $c > 0$.

   Given a family $\mathcal{R}$ of n rectangles in the plane with sides parallel to the axes, let $\tau(\mathcal{R})$ denote the *transversal number* of $\mathcal{R}$, i.e., the size of the smallest set of points with the property that every member of $\mathcal{R}$ contains at least one of them. It was conjectured by Gyárfás and Lehel [6] that $\tau(\mathcal{R}) \leq c'D(\mathcal{R})$, where c' is an absolute constant. Suppose for a moment that this conjecture is true. In view of the fact that $I(\mathcal{R}) \geq n/\tau(\mathcal{R})$, this implies $I(\mathcal{R})D(\mathcal{R}) \geq n/c'$. Hence $r(n) \geq \frac{1}{\sqrt{c'}}\sqrt{n}$, as required.

Theorem 2 can be easily generalized to higher dimensions.

Theorem 3 *Let $r_d(n)$ denote the maximum number with the property that given a family of n boxes in d-space with sides parallel to the axes, one can always choose $r_d(n)$ of them which are either pairwise disjoint or pairwise intersecting. Then*

$$c_d\sqrt{\frac{n}{\log_2^{d-1} n}} \leq r_d(n),$$

for a suitable constant $c_d > 0$. $\square$

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